Section 4: First Order Equations: Linear

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{1}$$

We'll introduce a new variable u, solve for u, then go back to the solution y.

Set $u = y^{1-n}$. Then $\frac{du}{dx} = (1-n)y^{1-n-1} \cdot \frac{dy}{dx}$

$$= (1-n)y^{-n}\frac{dy}{dx}$$

Since $1-n \neq 0$, we get $\frac{dy}{dx} = \frac{1}{1-n}y^n\frac{du}{dx}$

Plug into the ODE

$$\frac{1}{1-n}y^n\frac{du}{dx} + P(x)y = f(x)y^n$$

Divide by $\frac{1}{1-n} y^n$

$$\frac{du}{dx} + (1-n) P(x) \frac{y}{y^n} = (1-n) f(x) \underline{\frac{y^n}{y^n}}$$

$$\underset{1}{\underbrace{}}$$

Note: $\frac{y}{y^n} = y^{1-n} = u$

We arrive at the linear equation in u

$$\frac{du}{dx} + (1-n) P(x) u = (1-n) f(x).$$

We solve this equation for u using an integrating factor.

Then get y since

$$y = u^{\frac{1}{1-n}}.$$

Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

Here, $n = 3$. Set $u = y^{1-3} = y^{-2}$.

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{-1}{2}y^3\frac{du}{dx}$$

Subbing $\frac{-1}{2}y^3\frac{du}{dx} - y = -e^{2x}y^3$

Divide by $\frac{-1}{2}y^3$

$$\frac{du}{dx} + 2\frac{y}{y^3} = 2e^{2x}\frac{y^3}{y^3}$$

$$\underset{y^{-2} = u}{\underbrace{}} \qquad \underset{1}{\underbrace{}}$$

The equation for u is

$$\frac{du}{dx} + 2u = 2e^{2x}$$

1st order linear in standard form.

$$P(x) = 2 \implies \mu = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$$

$$e^{2x}\left(\frac{du}{dx} + 2u\right) = 2e^{2x}\left(e^{2x}\right)$$

$$\frac{d}{dx}\left[e^{2x} u\right] = 2e^{4x}$$

$$\int \frac{d}{dx}\left[e^{2x} u\right] dx = \int 2e^{4x} dx$$

$$e^{2x} u = \frac{2}{4} e^{4x} + C$$

$$u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}} = \frac{1}{2}e^{2x} + Ce^{-2x}$$

Since $u = y^{-2}$, $y = u^{-\frac{1}{2}} = \frac{1}{\sqrt{u}}$

Hence
$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}.$$

We apply the condition $y(0) = 1$.

$$1 = \frac{1}{\sqrt{\frac{1}{2}e^{0} + Ce^{0}}}$$

$$1 = \frac{1}{\sqrt{\frac{1}{2} + C}} \qquad \Rightarrow \qquad \sqrt{\frac{1}{2} + C} = \frac{1}{1} = 1$$

$$\left(\sqrt{\frac{1}{2} + C}\right)^2 = 1^2 \qquad \Rightarrow \qquad \frac{1}{2} + C = 1 \qquad \Rightarrow \qquad C = 1 - \frac{1}{2} = \frac{1}{2}$$

The solution to the IVP is

$$y = \frac{1}{\sqrt{\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}}}$$

Clearing the fractions, this can be written

$$y = \frac{\sqrt{2}}{\sqrt{e^{2x} + e^{-2x}}}$$

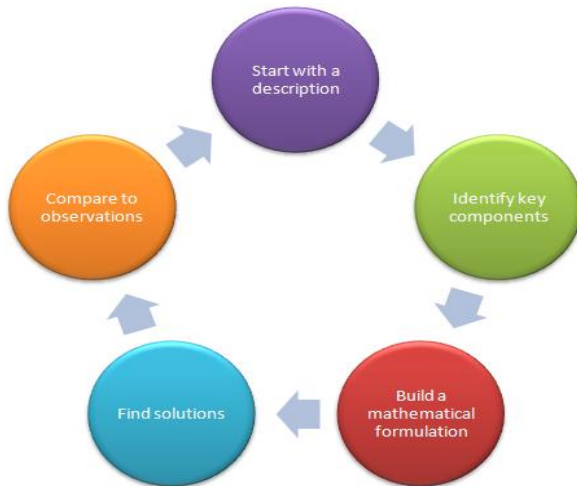# Section 5: First Order Equations Models and Applications

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let's derive an equation for population (density -ie. # rabbits per unit habitat). Let $P(t)$ be the population at time t. We'll take t in years with $t = 0$ in 2011.

Rate of change of $P = \dfrac{dP}{dt}$ proportional to P

$$\Rightarrow \quad \frac{dP}{dt} \propto P$$

i.e. $\qquad \dfrac{dP}{dt} = kP \qquad$ for some constant k.

This is a 1st order ODE for P, both separable and 1st order linear. From the statement, we have

$P(0) = 58$ and $P(1) = 89$.

Let's solve the IVP $\qquad \dfrac{dP}{dt} = kP$, $P(0) = 58$

Separate Variables $\qquad \dfrac{1}{P}\dfrac{dP}{dt} = k \implies \dfrac{1}{P}dP = k\,dt$

$$\int \dfrac{1}{P}dP = \int k\,dt \implies \ln|P| = kt + C$$

$$P = e^{kt+C} = e^{kt} \cdot e^{C} = Ae^{kt}, \quad A = e^{c}$$

$$P(0) = 58 \quad \text{so} \quad P(0) = Ae^{k \cdot 0} = 58 \implies A = 58$$

$$P(t) = 58\, e^{kt}. \quad \text{Using } P(1) = 89$$

$$58 e^{k \cdot 1} = 89 \implies e^{k} = \frac{89}{58} \implies k = \ln\left(\frac{89}{58}\right)$$

Hence $\quad P(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}. \quad$ In 2021, $\quad t = 10$

$$P(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)} \approx 4198$$

we expect almost 4200 rabbits by 2021.

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.