January 30 Math 2306 sec. 60 Spring 2018

Section 4: First Order Equations: Linear

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$

Steady and Transient States

Solution appears as $y = y_c + y_p$

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2e^{-x} + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2e^{-x} \quad \text{and} \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{1}$$

We'll define a new variable u, solve the equation for u, then find y from u. We set

$$u = y^{1-n}.$$

Differentiate $\quad \frac{du}{dx} = (1-n)\, y^{1-n-1} \frac{dy}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$

$1-n \neq 0$, so divide by $(1-n)\, y^{-n}$

$$\frac{dy}{dx} = \frac{1}{1-n}\, y^{n} \frac{du}{dx}$$

Sub into the ODE

$$\frac{1}{1-n} y^n \frac{du}{dx} + P(x)\, y = f(x)\, y^n$$

Multiply by $\dfrac{(1-n)}{y^n}$ $\qquad \dfrac{du}{dx} + (1-n)P(x)\underbrace{\dfrac{y}{y^n}}_{u} = (1-n) f(x) \underbrace{\dfrac{y^n}{y^n}}_{1}$

Note $\quad \dfrac{y}{y^n} = y^{1-n}$

The equation for u is

$$\frac{du}{dx} + (1-n)P(x)\, u = (1-n) f(x)$$

This is 1^{st} order linear. Solve by setting and integrating factor. Then from

$$u = y^{1-n}, \quad \text{we get}$$

$$y = u^{\frac{1}{1-n}}$$

Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

$n = 3$, so $u = y^{1-3} = y^{-2}$. Differentiate

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{-1}{2}y^3\frac{du}{dx}$$

Sub: $\frac{-1}{2}y^3\frac{du}{dx} - y = -e^{2x}y^3$

Mult. by $\frac{-2}{y^3}$

$$\frac{du}{dx} + 2\underbrace{\frac{y}{y^3}}_{y^{-2} = u} = 2e^{2x}\underbrace{\frac{y^3}{y^3}}_{1}$$

u solves $\quad \frac{du}{dx} + 2u = 2e^{2x} \quad$ It's in standard form

$P(x) = 2$, so $\quad \mu = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$

$$e^{2x}\left(\frac{du}{dx} + 2u\right) = 2e^{2x}\left(e^{2x}\right)$$

$$\frac{d}{dx}\left[e^{2x}u\right] = 2e^{4x}$$

$$\int \frac{d}{dx}\left[e^{2x}u\right]dx = \int 2e^{4x}\,dx$$

$$e^{2x}u = \frac{2}{4}e^{4x} + C$$

$$u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}} = \frac{1}{2}e^{2x} + Ce^{-2x}$$

$$u = y^{-2} \implies y = u^{-\frac{1}{2}} = \frac{1}{\sqrt{u}}$$

Hence $\quad y = \dfrac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}$

Now apply $y(0) = 1$

$$y(0) = 1 = \frac{1}{\sqrt{\frac{1}{2}e^{0} + Ce^{0}}}$$

$$\sqrt{\frac{1}{2}+C} = \frac{1}{1} = 1$$

$$\left(\sqrt{\frac{1}{2}+C}\right)^2 = 1^2 = 1$$

$$\frac{1}{2} + C = 1 \implies C = \frac{1}{2}$$

The solution to the IVP is

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}}}.$$

Clear the fractions

We can write this as

$$y = \frac{\sqrt{2}}{\sqrt{e^{2x} + e^{-2x}}}.$$

Section 5: First Order Equations Models and Applications

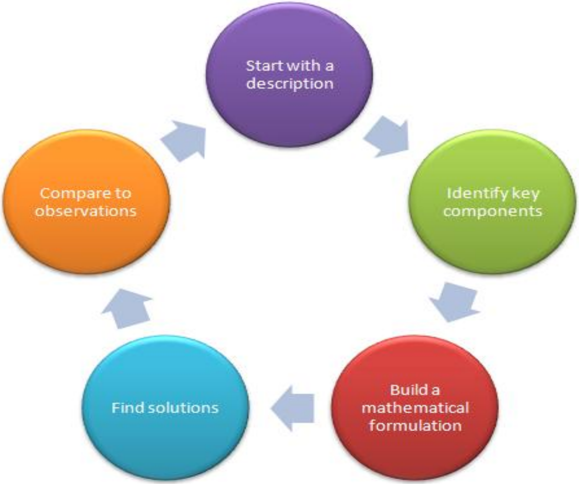

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let's let $P(t)$ be the rabbit population (density, # rabbits per unit area) at time t. Let's take t in years with $t=0$ in 2011.

instantaneous rate of change of $P = \frac{dP}{dt}$ is proportional to P.

That is $\frac{dP}{dt} \propto P$

Hence $\frac{dP}{dt} = kP$ for some constant k.

We also know that $P(0) = 58$ and $P(1) = 89$.

We can form an IVP by coupling the ODE with either condition.

We'll do this next time!