Section 1.4: The Matrix Equation $Ax = b$.

**Definition** Let $A$ be an $m \times n$ matrix whose columns are the vectors $a_1, a_2, \cdots, a_n$ (each in $\mathbb{R}^m$), and let $x$ be a vector in $\mathbb{R}^n$. Then the product of $A$ and $x$, denoted by

$$Ax$$

is the linear combination of the columns of $A$ whose weights are the corresponding entries in $x$. That is

$$Ax = x_1a_1 + x_2a_2 + \cdots + x_na_n.$$

(Note that the result is a vector in $\mathbb{R}^m$!)
Theorem

If $A$ is the $m \times n$ matrix whose columns are the vectors $a_1, a_2, \ldots, a_n$, and $b$ is in $\mathbb{R}^m$, then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[a_1 \ a_2 \ \cdots \ a_n \ b].$$
Corollary

The equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$.

In other words, the corresponding linear system is consistent if and only if $b$ is in $\text{Span}\{a_1, a_2, \ldots, a_n\}$.
Theorem (first in a string of equivalency theorems)

Let $A$ be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $b$ in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.

(b) Each $b$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.

(c) The columns of $A$ span $\mathbb{R}^m$.

(d) $A$ has a pivot position in every row.

(Note that statement (d) is about the coefficient matrix $A$, not about an augmented matrix $[A \ b]$.)
A Scalar Product

If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in \( \mathbb{R}^n \), we define a scalar product (also called the dot product) via

\[
\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n
\]

**Example** Compute \( \mathbf{u} \cdot \mathbf{v} \) if \( \mathbf{u} = (1, 2, 3) \) and \( \mathbf{v} = (-1, 0, 4) \).
Computing $Ax$

We can use a *row-vector* dot product rule. The $i^{th}$ entry in $Ax$ is the sum of products of corresponding entries from row $i$ of $A$ with those of $x$. For example

\[
\begin{bmatrix}
1 & 0 & -3 \\
-2 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix}
\]
\[
\begin{bmatrix}
2 & 4 \\
-1 & 1 \\
0 & 3 \\
\end{bmatrix}
\begin{bmatrix}
-3 \\
2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\]
Identity Matrix

We’ll call an $n \times n$ matrix with 1’s on the diagonal and 0’s everywhere else—i.e. one that looks like

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
$$

the $n \times n$ identity matrix and denote it by $I_n$. (We’ll drop the subscript if it’s obvious from the context.)

This matrix has the property that for each $x$ in $\mathbb{R}^n$

$$I_n x = x.$$
Theorem: Properties of the Matrix Product

If $A$ is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and $c$ is any scalar, then

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$. 
Section 1.5: Solution Sets of Linear Systems

**Definition** A linear system is said to be **homogeneous** if it can be written in the form

\[ Ax = 0 \]

for some \( m \times n \) matrix \( A \) and where \( 0 \) is the zero vector in \( \mathbb{R}^m \).

**Theorem:** A homogeneous system \( Ax = 0 \) always has at least one solution \( x = 0 \).

The solution \( x = 0 \) is called the **trivial solution**. A more interesting question for a homogeneous system is

**Does it have a nontrivial solution?**
Theorem
The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a) $2x_1 + x_2 = 0$
$x_1 - 3x_2 = 0$
\( \begin{align*}
3x_1 & + 5x_2 - 4x_3 = 0 \\
-3x_1 & - 2x_2 + 4x_3 = 0 \\
6x_1 & + x_2 - 8x_3 = 0
\end{align*} \)
(c) $x_1 - 2x_2 + 5x_3 = 0$
Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form \( x = x_3v \). Example (c)’s solution set consisted of vectors that look like \( x = x_2u + x_3v \). Since these are linear combinations, we could write the solution sets like

\[
\text{Span}\{u\} \quad \text{or} \quad \text{Span}\{u, v\}.
\]

Instead of using the variables \( x_2 \) and/or \( x_3 \) we often substitute parameters such as \( s \) or \( t \).

The forms

\[
x = su, \quad \text{or} \quad x = su + tv
\]

are called parametric vector forms.
Example

The **parametric vector form** of the solution set of
\[ x_1 - 2x_2 + 5x_3 = 0 \]

is

\[
\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.
\]

Question: What geometric object is that solution set?
Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

\[\begin{align*}
3x_1 + 5x_2 - 4x_3 &= 7 \\
-3x_1 - 2x_2 + 4x_3 &= -1 \\
6x_1 + x_2 - 8x_3 &= -4
\end{align*}\]
Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

\[ x = p + tv \]

with \( p \) and \( v \) fixed vectors and \( t \) a varying parameter. Also note that the \( tv \) part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

\( p \) is called a **particular solution**, and \( tv \) is called a solution to the associated homogeneous equation.
Theorem

Suppose the equation $Ax = b$ is consistent for a given $b$. Let $p$ be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the form

$$x = p + v_h,$$

where $v_h$ is any solution of the associated homogeneous equation $Ax = 0$.

We can use a row reduction technique to get all parts of the solution in one process.
Example

Find the solution set of the following system. Express the solution set in parametric vector form.

\[
\begin{align*}
  x_1 + x_2 - 2x_3 + 4x_4 &= 1 \\
  2x_1 + 3x_2 - 6x_3 + 12x_4 &= 4
\end{align*}
\]
Section 1.7: Linear Independence

We already know that a homogeneous equation $Ax = 0$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n]$ as

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n = 0.$$ 

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$. 
Definition: Linear Dependence/Independence

An indexed set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is said to be **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0
\]

has only the trivial solution.

The set \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exists a set of weights \( c_1, c_2, \ldots, c_p \) at least one of which is nonzero such that

\[
c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0.
\]

(i.e. Provided the homogeneous equation posses a nontrivial solution.)

An equation \( c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0 \), with at least one \( c_i \neq 0 \), is called a **linear dependence relation**.
Special Cases

A set with two vectors \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is linearly dependent if one is a scalar multiple of the other.
Example

Determine if the set is linearly dependent or linearly independent.

(a) \( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \)

(b) \( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \)
More than Two Vectors

Theorem: The columns of a matrix $A$ are linearly independent if and only if the homogeneous equation $Ax = 0$ has only the trivial solution.

Example: Determine if the set of vectors is linearly dependent or linearly independent. If they are dependent, find a linear dependence relation.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$
Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let $\mathbf{u}$ and $\mathbf{v}$ be any nonzero vectors in $\mathbb{R}^3$. Show that if $\mathbf{w}$ is any vector in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.
Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

Examine each set \(\{\mathbf{v}_1, \mathbf{v}_2\}\), \(\{\mathbf{v}_1, \mathbf{v}_3\}\), \(\{\mathbf{v}_2, \mathbf{v}_3\}\), and \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\).
Two More Theorems

**Theorem:** If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent**. That is, if \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \} \) is a set of vectors in \( \mathbb{R}^n \), and \( p > n \), then the set is linearly dependent.

**Theorem:** Any set of vectors that contains the zero vector is linearly dependent.
Determine if the set is linearly dependent or linearly independent

(a) \[ \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \]
Determine if the set is linearly dependent or linearly independent

\[
\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix} \right\}
\]
Determine if the set is linearly dependent or linearly independent

(c) \{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix} \}