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Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Definition Let A be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of A and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

is the linear combination of the columns of A whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

(Note that the result is a vector in $\mathbb{R}^m$!)

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

Corollary

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

In other words, the corresponding linear system is consistent if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

Last time, we did row reduction on the matrix $[A\ \mathbf{b}]$ and found that the equation is solvable only for vectors $\mathbf{b}$ such that

$$-2b_1 + b_2 - 2b_3 = 0. \quad \Rightarrow \quad b_1 = \frac{1}{2}b_2 - b_3$$

We can state this result in terms of a subset of $\mathbb{R}^3$.

We can restate the set of vectors $\vec{b}$ as

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -b_3 \\ 0 \\ b_3 \end{bmatrix}$$

$$= b_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

b_2, b_3 are any real numbers

So our allowable vectors $\vec{b}$ are linear combinations of the vectors $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

We can say that $A\vec{x}=\vec{b}$ is consistent if

$\vec{b}$ is in $\text{Span}\left\{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Theorem (first in a string of equivalency theorems)

Let A be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

(Note that statement (d) is about the *coefficient* matrix A, not about an augmented matrix $[A \quad \mathbf{b}]$.)

A Scalar Product

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, we define a scalar product (also called the *dot* product) via

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example Compute $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (-1, 0, 4)$.

$$\vec{u} \cdot \vec{v} = 1(-1) + 2(0) + 3(4) = 11$$

Computing $A\mathbf{x}$

We can use a *row-vector* dot product rule. The i^{th} entry in $A\mathbf{x}$ is the sum of products of corresponding entries from row i of A with those of $\mathbf{x}$. For example

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1(2)+0(1)+(-3)(-1) \\ -2(2)+(-1)\cdot 1+4(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

2×3

$\mathbb{R}^3$

product in $\mathbb{R}^2$

1^{st} entry: Dot product of row 1 of A with $\vec{x}$

2^{nd} entry: Dot product of row 2 of A with $\vec{x}$.

$$\begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(-3)+4(2) \\ -1(-3)+1(2) \\ 0(-3)+3(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1\cdot x_1 + 0x_2 + 0x_3 \\ 0x_1 + 1\cdot x_2 + 0x_3 \\ 0x_1 + 0\cdot x_2 + 1\cdot x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Identity Matrix

We'll call an $n \times n$ matrix with 1's on the diagonal and 0's everywhere else—i.e. one that looks like

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

the $n \times n$ **identity** matrix and denote it by I_n. (We'll drop the subscript if it's obvious from the context.)

This matrix has the property that for each $\mathbf{x}$ in $\mathbb{R}^n$

$$I_n\mathbf{x} = \mathbf{x}.$$

Theorem: Properties of the Matrix Product

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is any scalar, then

(a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$.

Section 1.5: Solution Sets of Linear Systems

Definition A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorem: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution $\mathbf{x} = \mathbf{0}$.

The solution $\mathbf{x} = \mathbf{0}$ is called the **trivial solution**. A more interesting question for a homogeneous system is

Does it have a nontrivial solution?

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a) $$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 0 \\ x_1 & - & 3x_2 & = & 0 \end{array}$$

we can state this as $A\vec{x} = \vec{0}$

where $A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This has augmented matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

There are no free variables, hence there are no nontrivial solutions.

The eqns read as $x_1 = 0$
$x_2 = 0$

The only solution is $\vec{x} = \vec{0}$.

Since the system is homogeneous, we can do rref on the coefficient matrix.

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{(b)}\quad \begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

An augmented matrix is

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

non pivot column

With a free variable, the system has nontrivial solutions.

From the rref

$x_1 = \frac{4}{3}x_3$

$x_2 = 0$

x_3 - free

All nontrivial solutions have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

In fact, all solutions are of the form

$$\vec{x} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix},$$ x_3 any real number.

(c) $x_1 - 2x_2 + 5x_3 = 0$

This has coefficiant matrix $\begin{bmatrix} 1 & -2 & 5 \end{bmatrix}$

It's already in rref. There are free variables, hence there are nontrivial solutions. $\vec{x}$ has to satisfy

$$x_1 = 2x_2 - 5x_3$$
$$x_2, x_3 - \text{free}$$

As a vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

x_2, x_3 any reals

Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form $\mathbf{x} = x_3\mathbf{v}$. Example (c)'s solution set consisted of vector that look like $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$. Since these are **linear combinations**, we could write the solution sets like

$$\text{Span}\{\mathbf{u}\} \quad \text{or} \quad \text{Span}\{\mathbf{u}, \mathbf{v}\}.$$

Instead of using the variables x_2 and/or x_3 we often substitute **parameters** such as s or t.
The forms

$$\mathbf{x} = s\mathbf{u}, \quad \text{or} \quad \mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

are called **parametric vector forms.**