January 31 MATH 1112 sec. 54 Spring 2020

Exponential Functions

Definition: Let a be a positive real number different from 1—i.e. $a > 0$ and $a \neq 1$. The function

$$f(x) = a^x$$

is called the **exponential function of base** a. Its domain is $(-\infty, \infty)$, and its range is $(0, \infty)$.

Some examples of exponential functions are

$$f(x) = 2^x, \qquad g(x) = \left(\frac{1}{3}\right)^x, \quad \text{and} \quad h(x) = \pi^{x-1}$$

Observation

We don't want to confuse exponential and power functions. Note that in an **exponential** function

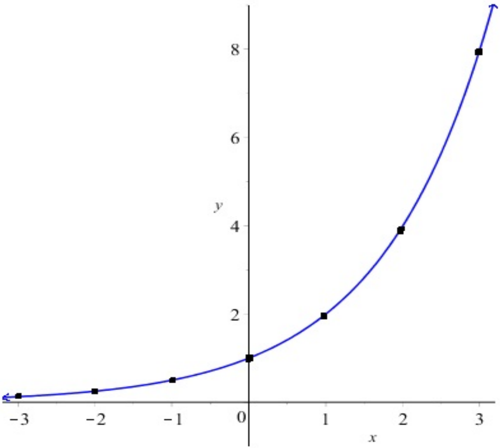
$$f(x) = 4^x$$

the base is a constant, and the exponent is a variable. Contrast a **power** function

$$f(x) = x^4$$

in which the base is variable, and the exponent is a constant.

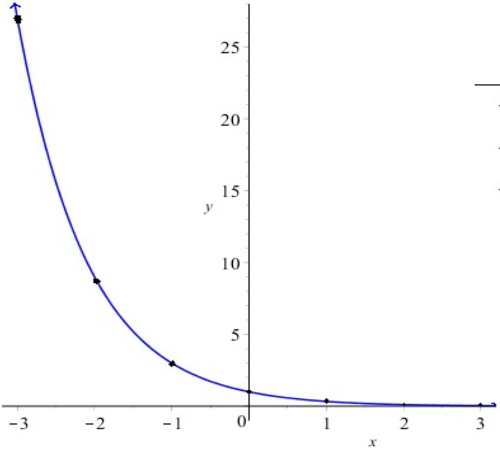
Graphs of Exponential Functions



x	$f(x) = 2^x$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

Figure: $f(x) = 2^x$ Note that the function is everywhere increasing. The x-axis is a horizontal asymptote.

Graphs of Exponential Functions



x	$f(x) = (1/3)^x$
-3	27
-2	9
-1	3
0	1
1	$1/3$
2	$1/9$
3	$1/27$

Figure: $f(x) = \left(\frac{1}{3}\right)^x$ Note that the function is everywhere decreasing. The x-axis is a horizontal asymptote.

Graphs of Exponential Functions

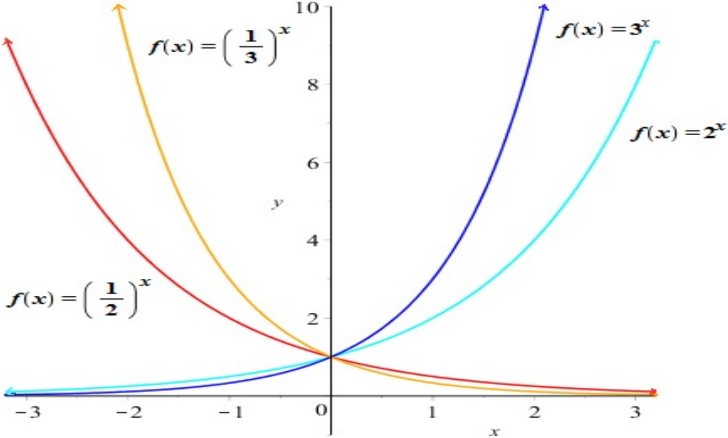


Figure: $f(x) = a^x$ is increasing if $a > 1$ and decreasing if $0 < a < 1$. The line $y = 0$ is a horizontal asymptote for every value of a. Each graph has y-intercept $(0, 1)$. Each graph is strictly above the x-axis.

a^x and a^{-x}

Let's observe that by properties of exponents, we have

$$f(x) = a^{-x} = \frac{1}{a^x}.$$

So as we saw suggested in the graphs, the plots of

$$f_1(x) = 2^x \quad \text{and} \quad f_2(x) = \left(\frac{1}{2}\right)^x$$

are reflections of one another in the y-axis.

The Favored Base

- From the graphs, we see that any base exponential can be obtained from any other base by stretching/shrinking and perhaps reflection in the y-axis.
- We can ask if there is a *natural* or prefered base.

The common base for *the* exponential function is the number

$$e \approx 2.718282.$$

The name e was given to this number by Leonhard Euler. It can be derived in several ways. One of these was discovered by Jacob Bernoulli in 1683 (this is credited as the first explicit derivation of the number).

e^x and e^{-x}

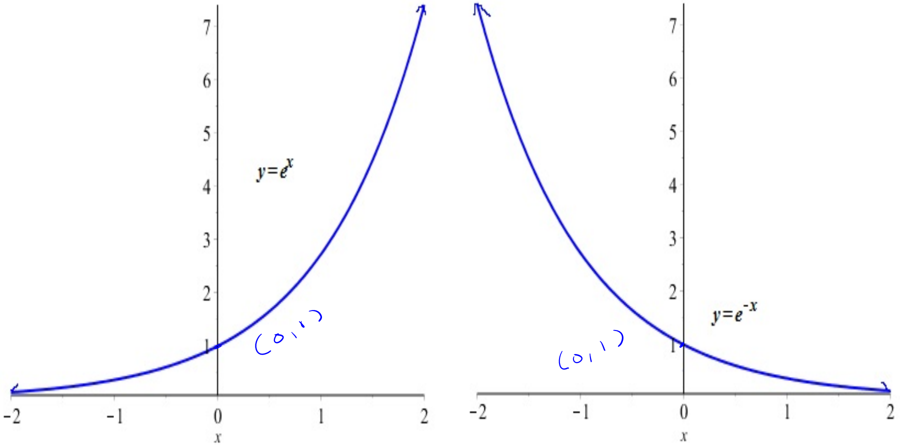


Figure: The exponential function $f(x) = e^x$ and the reciprocal function $g(x) = e^{-x}$ are among the most commonly used in applied mathematics. You should be able to produce these plots in your sleep!

Logarithms

Let's start with a **Question**.

True/False The exponential $f(x) = a^x$ is a one to one function.

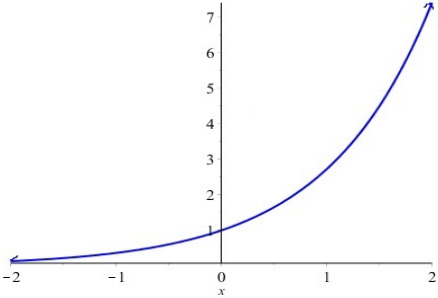


Figure: A plot of $f(x) = a^x$ for some $a > 1$.

Select (a) for True or (b) for False

Logarithms: Inverse of an Exponential Function

Since $f(x) = a^x$ is one to one with domain $(-\infty, \infty)$ and range $(0, \infty)$, there must be an inverse function f^{-1} with

- domain $(0, \infty)$,
- range $(-\infty, \infty)$, and such that
- $f^{-1}(x) = y$ if and only if $a^y = x$

For a given a, this inverse function is called the **logarithm function of base *a*.**

The Logarithm Function of Base a

Definition: Let $a > 0$ and $a \neq 1$. For $x > 0$ define $\log_a(x)$ as a number such that

$$\text{if} \quad y = \log_a(x) \quad \text{then} \quad x = a^y.$$

The function

$$F(x) = \log_a(x)$$

is called the **logarithm function of base** a. It has domain $(0, \infty)$, range $(-\infty, \infty)$, and if $f(x) = a^x$ then

$$F(x) = f^{-1}(x).$$

In particular

- $\log_a(a^x) = x$ for every real x, and
- $a^{\log_a(x)} = x$ for every $x > 0$.

Graph of Logarithms

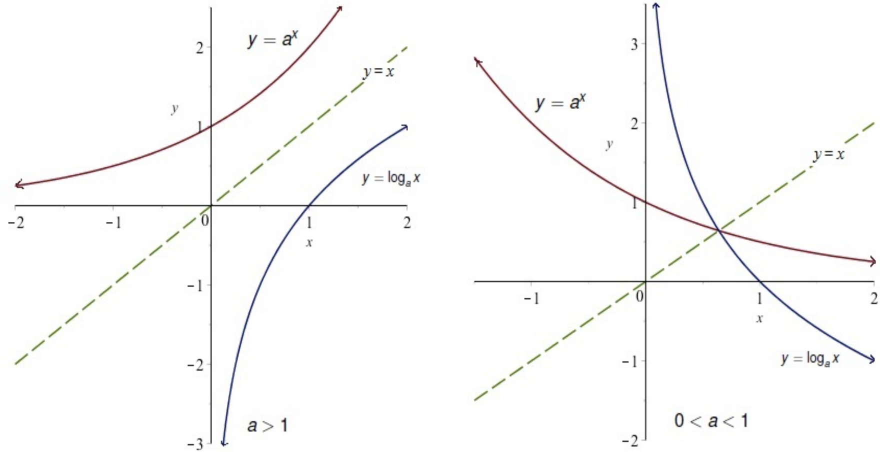


Figure: The graph of a logarithm can be obtained from the graph of an exponential by reflection in the line $y = x$. There are two cases depending on whether $0 < a < 1$ or $a > 1$.

Graph of Logarithms

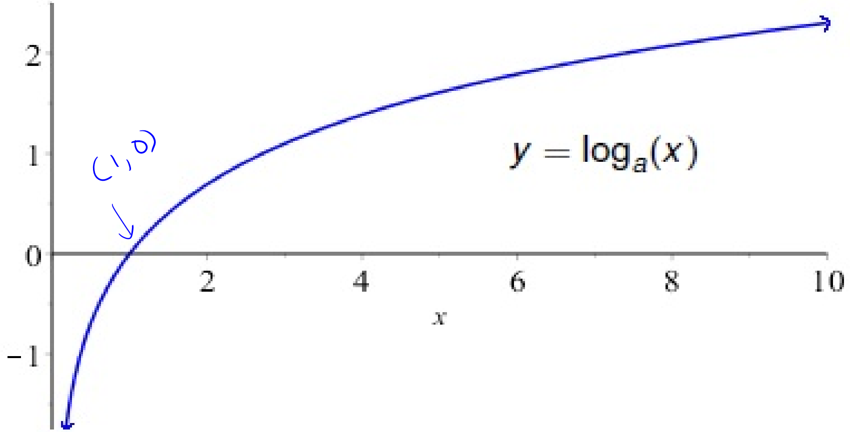


Figure: The graph of a logarithm with a base a where $a > 1$.

Graph of Logarithms

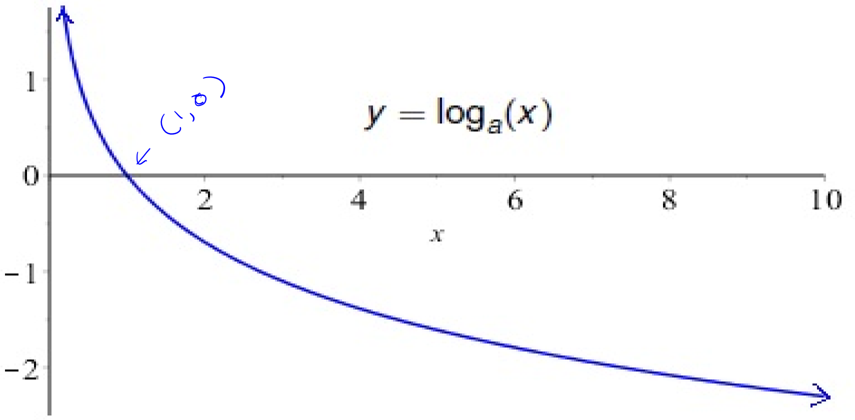


Figure: The graph of a logarithm with a base a where $0 < a < 1$

Evaluating Simple Logarithms

Use the fact that $y = \log_a(x)$ means $x = a^y$ to evaluate

(a) $\log_2(16)$ $= 4$

$\log_2(16) = 4 \iff 2^4 = 16$

(b) $\log_{10}(0.001)$ $= -3$

$\log_{10}(0.001) = -3 \iff 10^{-3} = 0.001$

(c) $\log_{1/2}(4)$ $= -2$

$\log_{\frac{1}{2}}(4) = -2$ $\quad \frac{1}{2}^{+2} = 4$