January 31 Math 3260 sec. 51 Spring 2020

Section 1.7: Linear Independence

Definition: An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \dots, c_p$ *at least one of which is nonzero* such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

(i.e. Provided the homogeneous equation posses a nontrivial solution.)

An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

Three Theorems on Linear Independence

Theorem: The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Theorem: If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent**. That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

Theorem: Any set of vectors that contains the zero vector is linearly **dependent**.

Determine if the set is linearly dependent or linearly independent

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

4 vectors in $\mathbb{R}^3$ $\quad 4 > 3$

They are lin. dependent.

Determine if the set is linearly dependent or linearly independent

$\vec{0}$

(b) $$\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}, \right\}$$

Set contains the zero vector. It is linearly dependent.

Section 1.8: Intro to Linear Transformations

Recall that the product $A\mathbf{x}$ is a linear combination of the columns of A—turns out to be a vector. If the columns of A are vectors in $\mathbb{R}^m$, and there are n of them, then

- A is an $m \times n$ matrix,
- the product $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbb{R}^n$, and
- the vector $\mathbf{b} = A\mathbf{x}$ is a vector in $\mathbb{R}^m$.

So we can think of A as an **object that acts** on vectors $\mathbf{x}$ in $\mathbb{R}^n$ (via the product $A\mathbf{x}$) to produce vectors $\mathbf{b}$ in $\mathbb{R}^m$.

Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition: A transformation T (a.k.a. **function** or **mapping**) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Function Notation: If a transformation T takes a vector $\mathbf{x}$ in $\mathbb{R}^n$ and maps it to a vector $T(\mathbf{x})$ in $\mathbb{R}^m$, we can write

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

which reads "T maps $\mathbb{R}^n$ into $\mathbb{R}^m$."

And we can write

$$\vec{x} \mapsto T(\mathbf{x})$$

which reads "$\mathbf{x}$ maps to T of $\mathbf{x}$."

Terms and Notation

For $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

- $\mathbb{R}^n$ is the **domain**, and
- $\mathbb{R}^m$ is called the **codomain.**
- For $\mathbf{x}$ in the domain, $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ under T. (We can call $\mathbf{x}$ a **pre-image** of $T(\mathbf{x})$.)
- The collection of all images is called the **range.**
- If $T(\mathbf{x})$ is defined by multiplication by the $m \times n$ matrix A, we may denote this by $\mathbf{x} \mapsto A\mathbf{x}$.

Matrix Transformation Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$. Define the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by the mapping $T(\mathbf{x}) = A\mathbf{x}$.

(a) Find the image of the vector $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ under T.

$$T(\vec{u}) = A\vec{u}$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1-9 \\ 2-12 \\ 0+6 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$$

(b) Determine a vector $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$.

Find $\vec{x}$ such that $T(\vec{x}) = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$

Solve $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$.

Using an augmented matrix

$$[A \ \vec{b}] = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2$

$x_2 = -2$

A pre-image $\vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$$

(c) Determine if $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in the range of T.

Is $T(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ solvable? $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

we can use an augmented matrix

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot column!

$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is inconsistent

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not in the range of T.

Linear Transformations

Definition: A transformation T is **linear** provided

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

Every matrix transformation (e.g. $\mathbf{x} \mapsto A\mathbf{x}$) is a linear transformation. And it turns out that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be expressed in terms of matrix multiplication.

A Theorem About Linear Transformations:

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0},$$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for scalars c, d and vectors $\mathbf{u}.\mathbf{v}$.

And in fact

$$T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \cdots + c_kT(\mathbf{u}_k).$$

Example

Let r be a nonzero scalar. The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = r\mathbf{x}$$

is a linear transformation[1].

Show that T is a linear transformation.

We have to show that $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

and $T(c\vec{u}) = c\,T(\vec{u})$

Note $T(\vec{u} + \vec{v}) = r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v} = T(\vec{u}) + T(\vec{v})$

$$T(c\vec{u}) = r(c\vec{u}) = rc\vec{u} = cr\vec{u} = c(r\vec{u}) = c\,T(\vec{u})$$

[1] It's called a **contraction** if $0 < r < 1$ and a **dilation** when $r > 1$

These follows from the algebraic properties of scalar multiplication.

Hence T is a linear transformation.

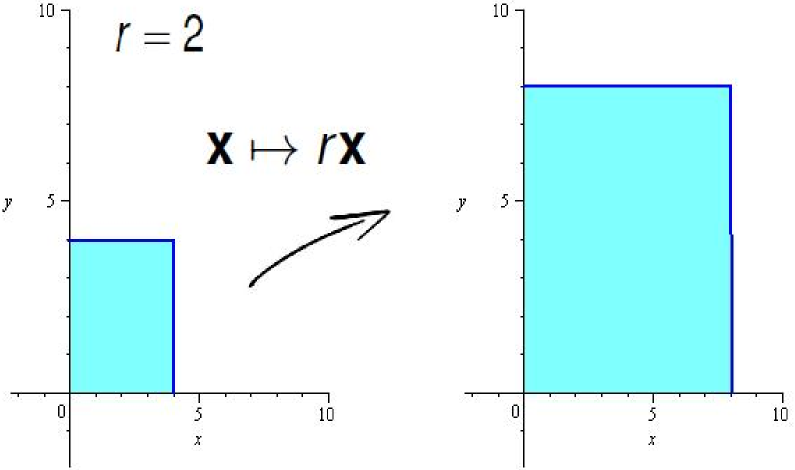

Figure: Geometry of dilation $\mathbf{x} \mapsto 2\mathbf{x}$. The 4 by 4 square maps to an 8 by 8 square.