Jan. 5 Math 2254H sec 015H Spring 2015

Section 6.1: Inverse Functions

Definition: A function f is called **one-to-one** if

$$x_1 \neq x_2 \quad \text{implies} \quad f(x_1) \neq f(x_2).$$

A one-to-one function can't take on the same value twice!

e.g. $f(x) = x^2$ is **NOT** one-to-one because $f(1) = f(-1)$ even though $1 \neq -1$.

e.g. $f(x) = x^3$ **IS** one-to-one because the only way $x_1^3 = x_2^3$ is if $x_1 = x_2$.

Horizontal Line Test

A function f is one-to-one if and only if no horizontal line intersects the graph $y = f(x)$ more than one time.

Recall that the vertical line test tells if a graph is that of a function. The horizontal line test can be used to tell if the graph of a function is that of a 1:1 function.

Graphically: One to one functions pass the Horizontal Line Test

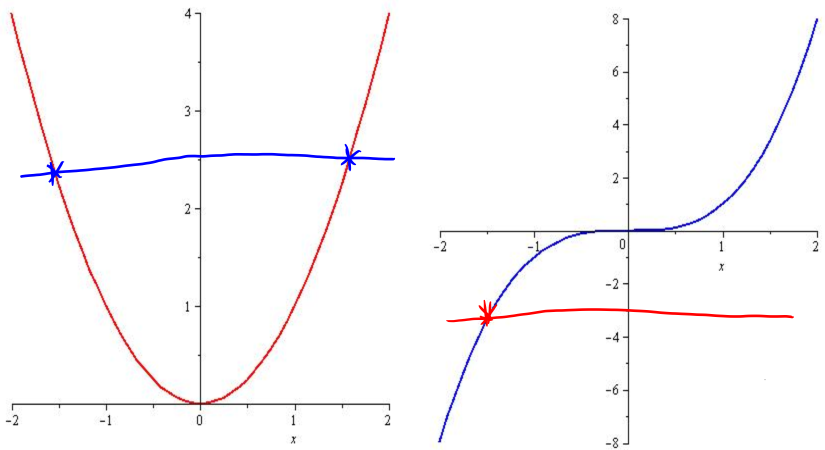


Figure: A function that is NOT 1:1 (left) and one that is 1:1 (right).

Theorem on one to one functions:

Theorem: If f is differentiable on an interval I, and $f'(x) > 0$ for all x in I, then f is one to one on I.

Question: If we change the condition $f'(x) > 0$ to read $f'(x) < 0$, can we still conclude that f would be 1:1?

Can f be one to one but have $f'(x_0) = 0$ for some x_0?

yes, take $f(x) = x^3$

Exercise: Prove the previous theorem by contradition.

<u>Proof</u>: Suppose $f'(x) > 0$ on an interval I. Also assume that f is not 1:1 on I.

Then there exist two numbers $a \neq b$ in I such that $f(a) = f(b)$. Assume $a < b$.

Note f is continuous on $[a,b]$ and differentiable on (a,b). By Rolle's theorem there exists a number c in (a,b) such that

$f'(c)=0$. But $f'(c)>0$ by our hypothesis. It must hold that no such a and b exist.!

Inverse Functions

Definition: Let f be a one-to-one function with domain A and range B. Then f has an inverse function $f^{-1}(x)$ defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y.$$

Remarks: (1) The domain of f^{-1} is B (the range of f).
(2) The range of f^{-1} is A (the domain of f).
(3) The notation f^{-1} **does not mean "reciprocal".**
(4) The following relationships hold:

$$f\left(f^{-1}(x)\right) = x \quad \text{for each } x \text{ in } B\text{, and}$$

$$f^{-1}\left(f(x)\right) = x \quad \text{for each } x \text{ in } A.$$

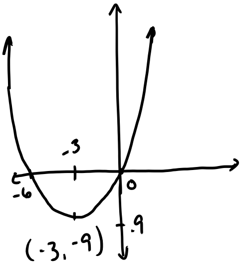
Example

Identify the range and find the inverse function f^{-1}. Verify one of the properties in (4), for

$$f(x) = x^2 + 6x, \quad \text{where} \quad A = \{x | x \geq -3\}.$$

$y = x^2 + 6x$

$= x(x+6)$



$f(-3) = 9 - 18 = -9$

The range is

$\{y \mid y \geq -9\}$

Find f^{-1}:
- Set $y = f(x)$
- Solve for x
- Swap variable names $x \longleftrightarrow y$

$y = x^2 + 6x \quad , \qquad x \geq -3 \qquad y \geq -9$

$y = x^2 + 6x + 9 - 9 \quad \Rightarrow \quad y + 9 = (x+3)^2$

$\Rightarrow \quad x + 3 = \sqrt{y+9} \quad \text{or} \quad x + 3 = -\sqrt{y+9}$

$x = \sqrt{y+9} - 3 \quad \text{or} \quad \cancel{x = -\sqrt{y+9} - 3}$

eliminate since $x \geq -3$

Swap variable names

$y = \sqrt{x+9} - 3$

This defines f^{-1}

$$\boxed{f^{-1}(x) = \sqrt{x+9} - 3, \qquad x \geq -9}$$

$$f(x) = x^2 + 6x, \qquad x \geq -3$$

Composition:

$$f^{-1}(f(x)) = f^{-1}(x^2+6x)$$

$$= \sqrt{x^2+6x+9} - 3$$

$$= \sqrt{(x+3)^2} - 3$$

$$= |x+3| - 3, \qquad x \geq -3 \Rightarrow |x+3| = x+3$$

$$= x+3-3 = x \quad \text{as required.}$$