# Jan. 6 Math 2254H sec 015H Spring 2015

## Section 6.1: Inverse Functions

**Definition:** A function $f$ is called **one-to-one** if

$$x_1 \neq x_2 \quad \text{implies} \quad f(x_1) \neq f(x_2).$$

**Definition:** Let $f$ be a one-to-one function with domain $A$ and range $B$. Then $f$ has an inverse function $f^{-1}(x)$ defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y.$$

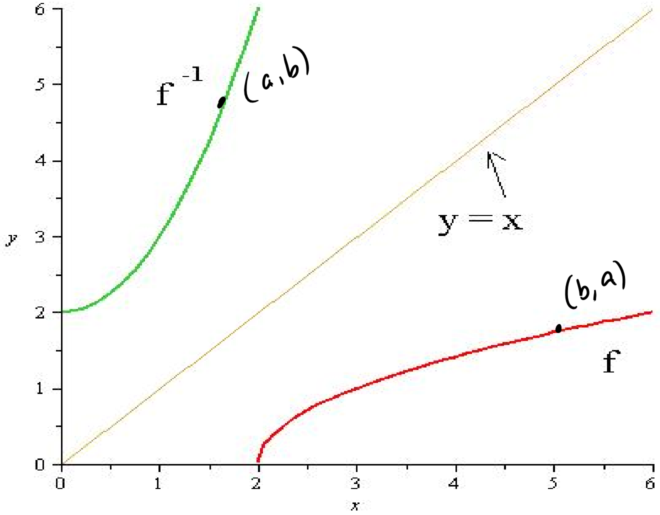



Figure: The graph of $f^{-1}$ is the reflection of the graph of $f$ in the line $y = x$.

## Continuity and Differentiability

**Theorem:** If $f$ is one-to-one and continuous on an interval, then its inverse $f^{-1}$ is continuous.

Inverses of continuous functions are continuous.

Next: Let $f$ be 1:1 with inverse $f^{-1}$ and suppose

$$f(b) = a \quad \text{i.e.} \quad f^{-1}(a) = b.$$

**Theorem:** If $f$ is differentiable at $b$, and $f'(b) \neq 0$, then $f^{-1}$ is differentiable at $a$ and

$$\left(f^{-1}\right)'(a) = \frac{1}{f'(b)}.$$

## Example

Consider

$$f(x) = x^2 + 6x, \quad x \geq -3 \quad \text{with} \quad f^{-1}(x) = \sqrt{x+9} - 3, \; x \geq -9.$$

Evaluate the two derivatives

$$f'(1), \quad \text{and} \quad \left(f^{-1}\right)'(7).$$

$f(1) = 1^2 + 6 \cdot 1 = 7$

$f'(x) = 2x + 6$ ,

$(f^{-1})'(x) = \frac{1}{2}(x+9)^{-1/2}$

$(f^{-1})'(x) = \dfrac{1}{2\sqrt{x+9}}$

$$f'(1) = 2 \cdot 1 + 6 = 8 \quad , \quad \left(f^{-1}\right)'(7) = \frac{1}{2\sqrt{7+9}} = \frac{1}{2\sqrt{16}} = \frac{1}{8}$$

# Plausibility Argument

Suppose $f'(x) \neq 0$, and consider the fact that $f^{-1}(f(x)) = x$. Use implicit differentiation to conclude that

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)}.$$

$f^{-1}(f(x)) = x$ Take $\frac{d}{dx}$ of both sides

$\frac{d}{dx}\left[f^{-1}(f(x))\right] = \frac{d}{dx}[x]$

$(f^{-1})'(f(x))\, f'(x) = 1$

$$\Longrightarrow \quad (f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

# Example

$$\left(f^{-1}\right)'(a) = \frac{1}{f'(b)} \quad \text{for} \quad f(b) = a$$

$f(x) = x^3 + 3\sin x + 2\cos x$. Find $\left(f^{-1}\right)'(2)$.

Need $b$ such that $f(b) = 2$

$$2 = b^3 + 3\sin(b) + 2\cos(b) \implies b = 0$$

by observation

$$f'(x) = 3x^2 + 3\cos x - 2\sin x$$

(It's not necessarily obvious, but $f$ is one-to-one.)

$$f'(0) = 3\cdot 0^2 + 3\cos(0) - 2\sin(0) = 3$$

$$\left(f^{-1}\right)'(2) = \frac{1}{f'(0)} = \frac{1}{3}$$

# Example

$$\left(f^{-1}\right)'(a) = \frac{1}{f'(b)} \quad \text{for } f(b) = a$$

$f(x) = \int_3^x 4^t \, dt$. Find $\left(f^{-1}\right)'(0)$.

Need $b$ such that $f(b) = 0$

$$0 = \int_3^b 4^t \, dt \implies b = 3$$

$$f'(x) = \frac{d}{dx} \int_3^x 4^t \, dt = 4^x$$

$$f'(3) = 4^3 = 64$$

$$\left(f^{-1}\right)'(0) = \frac{1}{f'(3)} = \frac{1}{64}$$

# Section 6.2*: The Natural Logarithm

We know that

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

provided $n \neq -1$. But we have yet to know how to understand

$$\int \frac{1}{x} \, dx.$$

We begin with a definite integral.

# Definition of the Natural Logarithm

Let $x > 0$. The natural logarithm of $x$ is denoted and define by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

By the **Fundamental Theorem of Calculus**, we immediately get

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Also, $\ln(1) = \int_1^1 \frac{dt}{t} = 0$.

# A Geometric Interpretation of the Nature Logarithm

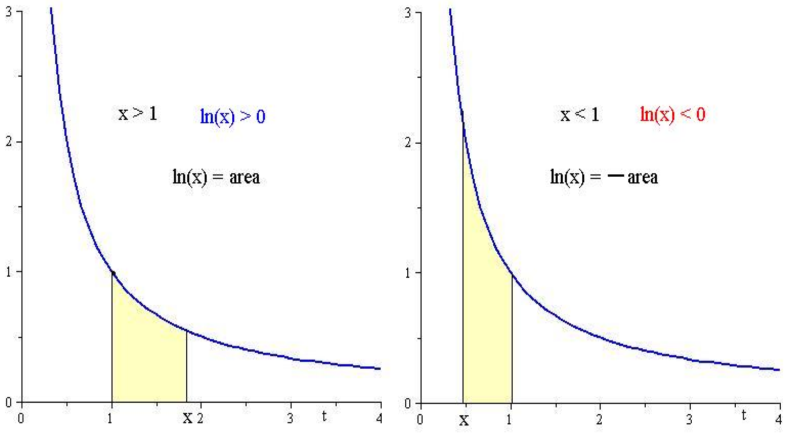



Figure: $\ln(1) = 0$, $\ln(x)$ is positive for $x > 1$ and $\ln(x)$ is negative for $0 < x < 1$.

# Properties of The Natural Log

Let $x$ and $y$ be positive real numbers and $r$ be a rational number.

$$(1)\quad \ln(xy) = \ln x + \ln y, \quad (2)\quad \ln\left(\frac{x}{y}\right) = \ln x - \ln y, \quad \text{and}$$

$$(3)\quad \ln(x^r) = r\ln x.$$

---

Example: Expand the expression $\ln\left[\frac{(x^3+2x)^4\cos(2x)}{\sqrt{x}\sin x}\right]$

$$= \ln\left[(x^3+2x)^4\cos(2x)\right] - \ln\left[\sqrt{x}\,\sin x\right]$$

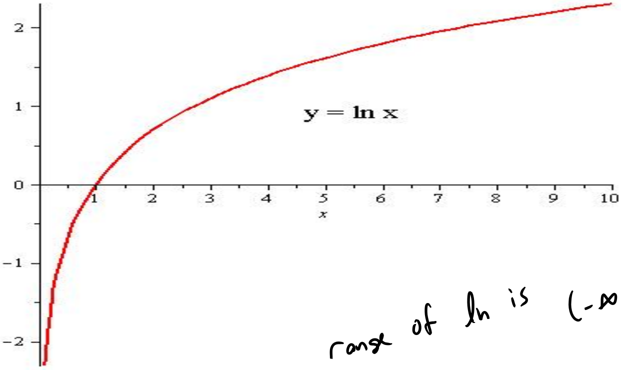
$$= \ln (x^3+2x)^4 + \ln(\cos(2x)) - \left(\ln\sqrt{x} + \ln(\sin x)\right)$$

$$= \ln (x^3+2x)^4 + \ln(\cos(2x)) - \ln x^{1/2} - \ln(\sin x)$$

$$= 4 \ln (x^3+2x) + \ln(\cos(2x)) - \frac{1}{2}\ln x - \ln(\sin x)$$

# Properties of The Natural Log

$$\lim_{x\to 0^+} \ln x = -\infty, \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty.$$




range of ln is $(-\infty, \infty)$

Figure: Plot of the natural log function.

# Definition: The number $e$

$e$ is the number such that $\ln e = 1$

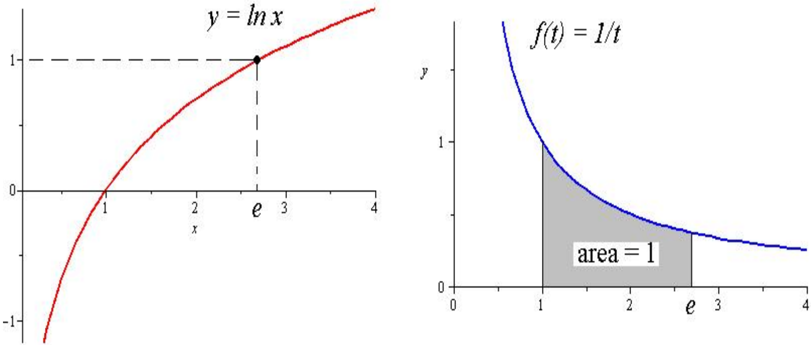



Figure: The number $e \approx 2.71828183$

# Differentiation & Integration Rules

$$(1) \quad \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx} \quad \text{i.e.} \quad \frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$$

$$(2) \quad \frac{d}{dx}\ln|x| = \frac{1}{x}, \quad \text{and}$$

$$(3) \quad \int \frac{dx}{x} = \ln|x| + C$$

If $x<0$, then $|x| = -x$. So $\ln|x| = \ln(-x)$

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$$

## Examples

Evaluate the derivative.

(a) $\frac{d}{dx}\ln(x^7+2x^3+2) = \dfrac{7x^6+6x^2}{x^7+2x^3+2}$

(b) $\frac{d}{dx}\ln(\sec x) = \dfrac{\sec x \tan x}{\sec x} = \tan x$

This implies $\int \tan x \, dx = \ln|\sec x| + C$

Evaluate

(a) $\displaystyle\int \frac{x}{x^2+5}\,dx$

Set $u = x^2+5$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$= \displaystyle\int \frac{\frac{1}{2}\,du}{u}$

$= \frac{1}{2}\displaystyle\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2+5| + C$