January 7 Math 2306 sec. 60 Spring 2019

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact.

$$\frac{dy}{dx} = -2\sin(2x).$$

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Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$.

Suppose $y = \cos(2x)$

Note that
$$\frac{d^2y}{dx^2} + 4y = 0.$$

We know that $y'' = -4 \cos(2x)$. So
 $y'' + 4y = -4 \cos(2x) + 4 \cos(2x) = 0$

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A differential equation

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a differential equation.

Questions: If we only started with the equation, how could we determine that cos(2x) satisfies it? Also, is cos(2x) the only possible function that *y* could be?



A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

with respect to.

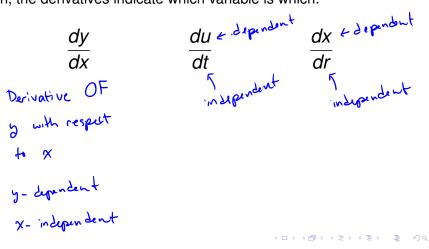
Dependent Variable: will appear as one that derivatives are taken of.

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Independent and Dependent Variables

Often, the derivatives indicate which variable is which:



Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable¹. For example

$$\frac{dy}{dx} - y^2 = 3x$$
, or $\frac{dy}{dt} + 2\frac{dx}{dt} = t$, or $y'' + 4y = 0$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

¹These are the subject of this course.

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Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

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 $\frac{dy}{dx} - y^2 = 3x \qquad 1^{st} \text{ order } ODE$ $y''' + (y')^4 = x^3 \qquad 3^{rd} \text{ order } ODE$ $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \qquad \partial^{nd} \text{ order } DDE$

Notations and Symbols

We'll use standard derivative notations:

Leibniz:
$$\frac{dy}{dx}$$
, $\frac{d^2y}{dx^2}$, ... $\frac{d^ny}{dx^n}$, or
Prime & superscripts: y' , y'' , ... $y^{(n)}$.

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

velocity is
$$\frac{ds}{dt} = \dot{s}$$
, and acceleration is $\frac{d^2s}{dt^2} = \ddot{s}$

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Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where *F* is some real valued function of n + 2 variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

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$$\frac{d^2y}{dx^2} + 4y = 0$$
This has the form $F(x,y,y',y'') = 0$
where $F(x,y,y',y'') = y'' + 4y$
In normal form, $\frac{d^2y}{dx^2} = f(x,y,y')$

$$\frac{d^2y}{dx^2} = -4y$$
so $f(x,y,y') = -4y$

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Notations and Symbols

If n = 1 or n = 2, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y)$$
 or $\frac{d^2y}{dx^2} = f(x, y, y').$

Differential Form: A first order equation may appear in the form

$$M(x, y) dx + N(x, y) dy = 0$$
This is a differential form

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$$M(x,y)\,dx+N(x,y)\,dy=0$$

Differential forms may be written in normal form in a couple of ways.

If N(x,y) = 0 (for all relevant x and y), we can rearrange this as N(x, y) dy = - M(x, y) dx $\Rightarrow \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$ Similarly, if M(x,5) =0, we could write this $\frac{dx}{dx} = -\frac{N(x,y)}{N(x,y)}$ as イロト 不得 トイヨト イヨト 二日

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Classifications

Linearity: An *n*th order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients a_0, \ldots, a_n and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

Examples (Linear -vs- Nonlinear)

$$y'' + 4y = 0 \qquad t^2 \frac{d^2x}{dt^2} + 2t \frac{dx}{dt} - x = e^t$$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

$$u'' + u' = \cos u$$