January 7 Math 2306 sec. 60 Spring 2019

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$.

Suppose $y = \cos(2x)$

Note that $\dfrac{d^2y}{dx^2} + 4y = 0.$

we know that $y'' = -4\cos(2x)$. so

$y'' + 4y = -4\cos(2x) + 4\cos(2x) = 0$

A *differential* equation

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a **differential equation.**

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it? Also, is $\cos(2x)$ the only possible function that y could be?

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

← this is a function

Dependent Variable: will appear as one that derivatives are taken of.

$y = \cos(2x)$ Dependent y is a function of independent x.

Independent and Dependent Variables

Often, the derivatives indicate which variable is which:

$$\frac{dy}{dx} \qquad \frac{du}{dt} \leftarrow \text{dependent} \qquad \frac{dx}{dr} \leftarrow \text{dependent}$$

Derivative OF

y with respect

to x

independent

independent

y - dependent

x - independent

Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

[1] These are the subject of this course.

Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$ 1st order ODE

$$y''' + (y')^4 = x^3$$ 3rd order ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$ 2nd order PDE

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}).$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

This has the form $F(x,y,y',y'') = 0$

where $F(x,y,y',y'') = y'' + 4y$

In normal form, $\frac{d^2y}{dx^2} = f(x,y,y')$

$$\frac{d^2y}{dx^2} = -4y \quad \text{so} \quad f(x,y,y') = -4y$$

Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x, y, y').$$

Differential Form: A first order equation may appear in the form

$$\underbrace{M(x, y)\,dx + N(x, y)\,dy}_{\text{This is a differential form}} = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Differential forms may be written in normal form in a couple of ways.

If $N(x,y) \neq 0$ (for all relevant x and y), we can rearrange this as

$$N(x,y)\,dy = -M(x,y)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

Similarly, if $M(x,y) \neq 0$, we could write this as

$$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)}$$

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

Examples (Linear -vs- Nonlinear)

$$y'' + 4y = 0$$

$$t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

$$u'' + u' = \cos u$$