Jan. 8 Math 2254H sec 015H Spring 2015

Section 6.2*: The Natural Logarithm

Definition: Let $x > 0$. The natural logarithm of x is denoted and defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

We determined that

- $\ln(1) = 0$,
- $\ln(x) > 0$ for $x > 1$,
- $\ln(x) < 0$ for $0 < x < 1$, and
- $\frac{d}{dx}\ln(x) = \frac{1}{x}$

Properties of The Natural Log

$$\lim_{x\to 0^+} \ln x = -\infty, \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty.$$

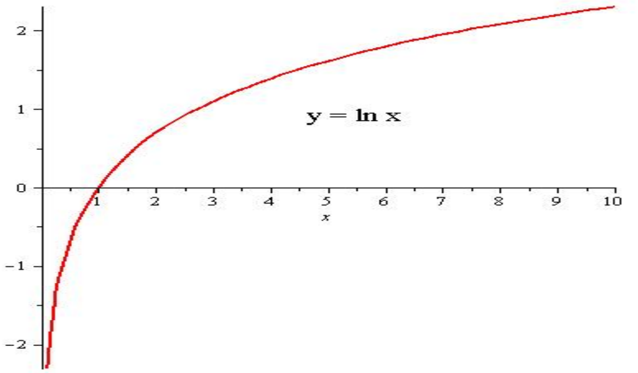

Figure: Plot of the natural log function.

Properties of The Natural Log

Let x and y be positive real numbers and r be a rational number.

$$(1) \quad \ln(xy) = \ln x + \ln y,$$

$$(2) \quad \ln\left(\frac{x}{y}\right) = \ln x - \ln y, \quad \text{and}$$

$$(3) \quad \ln(x^r) = r \ln x.$$

Differentiation & Integration Rules

$$(1) \quad \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx} \quad \text{i.e.} \quad \frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$$

$$(2) \quad \frac{d}{dx}\ln|x| = \frac{1}{x}, \quad \text{and}$$

$$(3) \quad \int \frac{dx}{x} = \ln|x| + C, \quad \text{in general} \quad \int \frac{du}{u} = \ln|u| + C$$

Examples

(b) $\int_e^{e^2} \frac{\ln x}{x}\, dx = \int_e^{e^2} (\ln x) \frac{1}{x}\, dx$

$u = \ln x$

$du = \frac{1}{x}\, dx$

when $x = e$

$u = \ln e = 1$

$x = e^2$

$u = \ln e^2$

$= 2 \ln e$

$= 2 \cdot 1 = 2$

$= \int_1^2 u\, du$

$= \frac{u^2}{2} \Big|_1^2$

$= \frac{2^2}{2} - \frac{1^2}{2} = 2 - \frac{1}{2} = \frac{3}{2}$

(c) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$

$u = \sin x$

$du = \cos x \, dx$

$= \int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|\sin x| + C$

New Integration Rules

$$\int \tan x\, dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$\int \cot x\, dx = \ln|\sin x| + C = -\ln|\csc x| + C$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Multiply the integrand by

$$1 = \frac{\sec x + \tan x}{\sec x + \tan x}$$

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

let

$$u = \sec x + \tan x$$

$$du = (\sec x \tan x + \sec^2 x) \, dx$$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Logarithmic Differentiation

Use the logarithm to evaluate $\frac{dy}{dx}$. Assume that $y > 0$ if necessary.

$$y = \frac{(x^3+2x)^3 \sin x}{\sqrt[3]{x}\cos^2 x}$$

Note: $\frac{d}{dx} \ln y = \frac{\frac{dy}{dx}}{y}$

i.e. $\frac{dy}{dx} = y \frac{d}{dx} \ln y$

Take the log

$$\ln y = \ln\left(\frac{(x^3+2x)^3 \sin x}{x^{1/3} \cos^2 x}\right)$$

$$= 3\ln(x^3+2x) + \ln \sin x - \frac{1}{3}\ln x - 2\ln \cos x$$

Take $\frac{d}{dx}$ of both sides

$$\frac{1}{y}\frac{dy}{dx} = 3\,\frac{3x^2+2}{x^3+2x} + \frac{\cos x}{\sin x} - \frac{1}{3}\,\frac{1}{x} - 2\left(\frac{-\sin x}{\cos x}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{9x^2+6}{x^3+2x} + \cot x - \frac{1}{3x} + 2\tan x$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^3+2x)^3 \sin x}{\sqrt[3]{x}\,\cos^2 x}\left(\frac{9x^2+6}{x^3+2x} + \cot x - \frac{1}{3x} + 2\tan x\right)$$

Section 6.3* The Natural Exponential Function

Remark: The natural logarithm is a one-to-one function.

Definition: The **exponential** function is the inverse of the natural logarithm. It is denoted by

$$\exp(x) \quad \text{or} \quad e^x$$

and defined by

$$\exp(x) = y \quad \text{if and only if} \quad x = \ln y.$$

Relationship between $\ln x$ and $\exp x$

- $\ln x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$
- $\exp x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$
- $\exp(\ln x) = x$ for all $x > 0$ and $\ln(\exp x) = x$ for all x
- Since $\ln x \to -\infty$ as $x \to 0^+$ we have
$$\lim_{x \to -\infty} \exp(x) = 0,$$
- and since $\ln x \to \infty$ as $x \to \infty$ we have
$$\lim_{x \to \infty} \exp(x) = \infty.$$

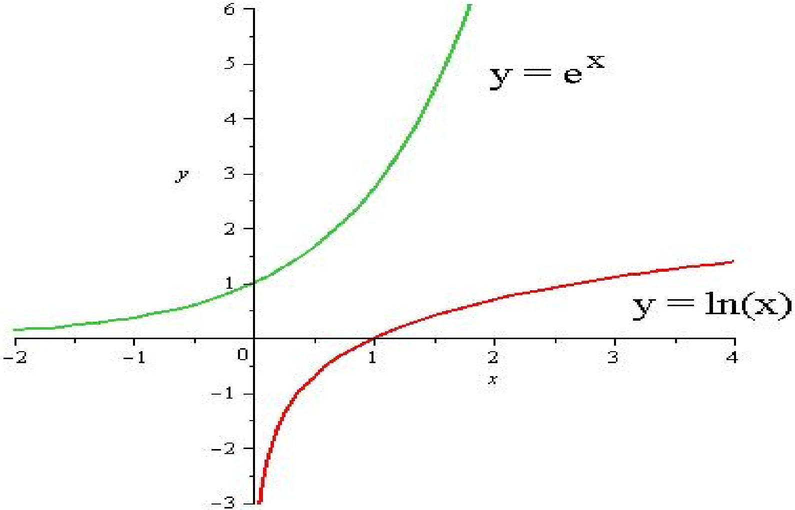

Figure: The graph of $y = e^x$ (with graph of $y = \ln x$ for comparison).

Laws of Exponents

Let x and y be real and r rational. Then

$$(1) \quad e^{x+y} = e^x e^y, \quad (2) \quad e^{x-y} = \frac{e^x}{e^y}, \quad \text{and}$$

$$(3) \quad (e^x)^r = e^{rx}.$$

Differentiation Rule[1]

$$\frac{d}{dx} e^x = e^x, \quad \text{so that} \quad \frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

[1]The function $f(x) = e^x$ is differentiable for all real x, hence it is continuous at all real x.