January 8 Math 3260 sec. 51 Spring 2020

Section 1.1: Systems of Linear Equations

We defined the following:

- Linear equation and linear system,
- solution, and solution set, and
- Equivalent systems.

An Example

$$\begin{array}{rcrcr} 2x & - & y & = & -1 \\ -4x & + & 2y & = & 2 \end{array}$$

For this system, we saw that the ordered pair $(1, 3)$ is a solution. The solution set is $\{(x, y) | y = 2x + 1\}$. So this is an example of a system that has a solution, but has more than one. Turns out, we can make some observations and generalize.

The Geometry of 2 Equations with 2 Variables

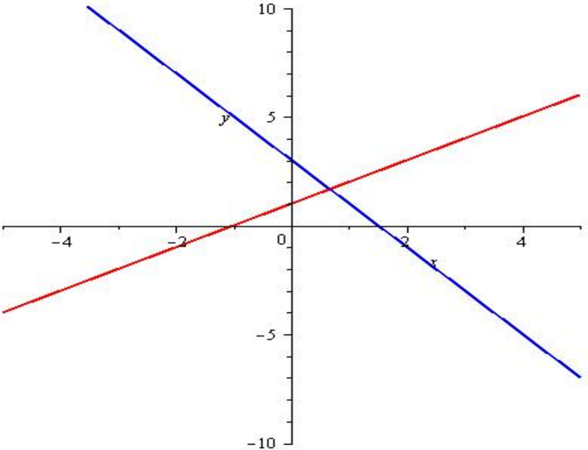

Figure: The system $x - y = -1$ and $2x + y = 3$ with solution set $\{(2/3, 5/2)\}$. These equations represent lines that intersect at one point.

The Geometry of 2 Equations with 2 Variables

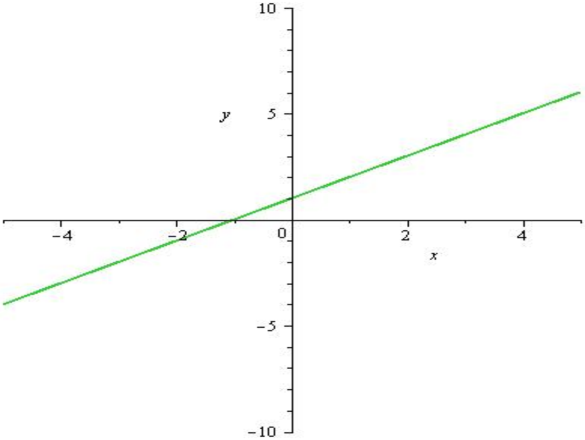

Figure: The system $x - y = -1$ and $2x - 2y = -2$ with solution set $\{(x, y) | y = x + 1\}$. Both equations represent the same line which share all common points as solutions.

The Geometry of 2 Equations with 2 Variables

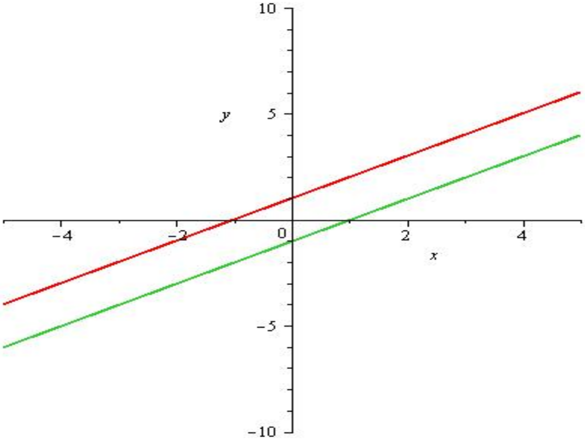

Figure: The system $x - y = -1$ and $2x - 2y = 2$ with solution set $\emptyset$. These equations represent parallel lines having no common points.

Theorem

A linear system of equations has exactly one of the following:

i No solution, or

ii Exactly one solution, or

iii Infinitely many solutions.

Terms: A system is
consistent if it has at least one solution (cases ii and iii), and
inconsistent if is has no solutions (case i).

Recall those two critical questions about a linear system: (1) Does it have a solution? (existence), and (2) If it has a solution, is there only one? (uniqueness)

Legitimate Operations for Solving a System

We can perform three basic operation **without** changing the solution set of a system. These are

- swap the order of any two equations **(swap)**,
- multiply an equation by any nonzero constant **(scale)**, and
- replace an equation with the sum of itself and a nonzero multiple of any other equation **(replace)**.

We'll try to solve a system by using these operations to *eliminate* variables from equations.

Some Operation Notation

E_i is the ith equation

Notation

- Swap equations i and j:
$$E_i \leftrightarrow E_j$$
- Scale equation i by k:
$$kE_i \rightarrow E_i$$
- Replace equation j with the sum of itself and k times equation i:
$$kE_i + E_j \rightarrow E_j$$

Solve the following system of equations by *elimination*[1].

$-2E_1 + E_2$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$$\begin{array}{rcrcr} -2x_1 & -4x_2 & +2x_3 & = & 8 \\ 2x_1 & & +x_3 & = & 7 \end{array}$$

First, we'll *eliminate* x_1 from the second and third equation.

$$-2E_1 + E_2 \longrightarrow E_2$$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & -4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

[1] The process here is technically called *Gaussian Elimination*.

$-E_1 + E_3$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & - & 4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$-x_1 - 2x_2 + x_3 = 4$

$x_1 + x_2 + x_3 = 6$

$$-E_1 + E_3 \longrightarrow E_3$$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & - & 4x_2 & + & 3x_3 & = & 15 \\ & - & x_2 & + & 2x_3 & = & 10 \end{array}$$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & - & 4x_2 & + & 3x_3 & = & 15 \\ & - & x_2 & + & 2x_3 & = & 10 \end{array}$$

Now, we eliminate x_2 from the third equation (without reintroducing x_1). One option is $-\frac{1}{4}E_2 + E_3 \longrightarrow E_3$. But to avoid fractions, let's swap first.

$$E_2 \leftrightarrow E_3$$

$$\begin{array}{l} x_1 + 2x_2 - x_3 = -4 \\ -x_2 + 2x_3 = 10 \\ -4x_2 + 3x_3 = 15 \end{array}$$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & - & x_2 & + & 2x_3 & = & 10 \\ & - & 4x_2 & + & 3x_3 & = & 15 \end{array}$$

$-4E_2 + E_3$

$4x_2 - 8x_3 = -40$

$-4x_2 + 3x_3 = 15$

Now to eliminate x_2: $\quad -4E_2 + E_3 \longrightarrow E_3$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & -x_2 & + & 2x_3 & = & 10 \\ & & & & -5x_3 & = & -25 \end{array}$$

$$
\begin{array}{rcrcrcr}
x_1 & + & 2x_2 & - & x_3 & = & -4 \\
 & - & x_2 & + & 2x_3 & = & 10 \\
 & & & - & 5x_3 & = & -25
\end{array}
$$

We can clean this up a little bit by performing

$$-E_2 \longrightarrow E_2 \quad \text{and} \quad -\frac{1}{5}E_3 \longrightarrow E_3$$

$$
\begin{array}{rcrcrcr}
x_1 & + & 2x_2 & - & x_3 & = & -4 \\
 & & x_2 & - & 2x_3 & = & -10 \\
 & & & & x_3 & = & 5
\end{array}
$$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & x_2 & - & 2x_3 & = & -10 \\ & & & & x_3 & = & 5 \end{array}$$

Now the solution can be obtained with a little (back) substitution.

From E_3, $x_3 = 5$

From E_2 $\quad x_2 = -10 + 2x_3 = -10 + 2(5)$

$= 0$

From E_1, $\quad x_1 = -4 - 2x_2 + x_3$

$= -4 - 2(0) + 5$

$= 1$.

One way to state the conclusion is

$$x_1 = 1$$

$$x_2 = 0$$

$$x_3 = 5$$

We could express the solution as

$$(1, 0, 5)$$