Math 2254H sec 015H Spring 2015

Section 6.3* The Natural Exponential Function

Remark: The natural logarithm is a one-to-one function.

Definition: The **exponential** function is the inverse of the natural logarithm. It is denoted by

$$\exp(x) \quad \text{or} \quad e^x$$

and defined by

$$\exp(x) = y \quad \text{if and only if} \quad x = \ln y.$$

Relationship between $\ln x$ and $\exp x$

- $\ln x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$
- $\exp x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$
- $\exp(\ln x) = x$ for all $x > 0$ and $\ln(\exp x) = x$ for all x
- Since $\ln x \to -\infty$ as $x \to 0^+$ we have

$$\lim_{x \to -\infty} \exp(x) = 0,$$

- and since $\ln x \to \infty$ as $x \to \infty$ we have

$$\lim_{x \to \infty} \exp(x) = \infty.$$

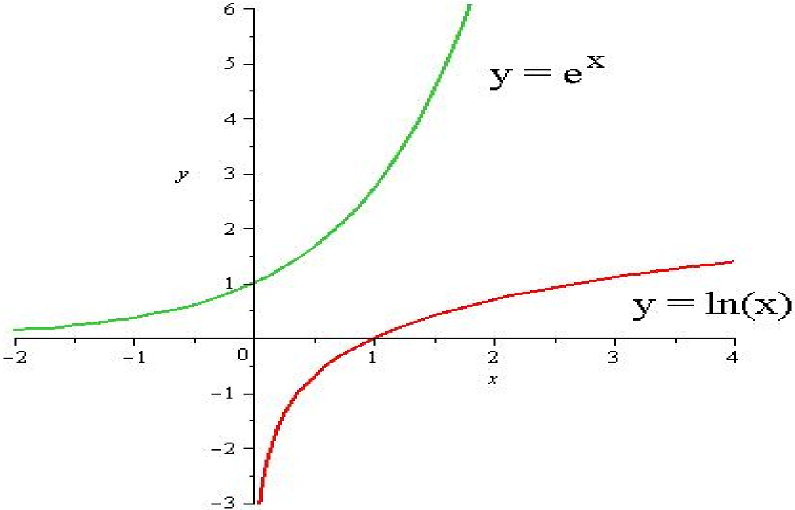

Figure: The graph of $y = e^x$ (with graph of $y = \ln x$ for comparison).

Laws of Exponents

Let x and y be real and r rational. Then

$$(1)\quad e^{x+y}=e^xe^y,\quad (2)\quad e^{x-y}=\frac{e^x}{e^y},\quad \text{and}$$

$$(3)\quad (e^x)^r=e^{rx}.$$

Differentiation Rule[1]

$$\frac{d}{dx}e^x=e^x,\quad \text{so that}\quad \frac{d}{dx}e^u=e^u\frac{du}{dx}.$$

[1]The function $f(x)=e^x$ is differentiable for all real x, hence it is continuous at all real x.

For all real x, $\ln(e^x) = x$. Show that it follows that $\frac{d}{dx} e^x = e^x$

$$\frac{d}{dx}\left[\ln(e^x) \right] = \frac{d}{dx}[x]$$

$$\frac{1}{e^x} \frac{d}{dx} e^x = 1 \quad \Rightarrow$$

$$\frac{d}{dx} e^x = e^x$$

Evaluate each derivative

(a) $\dfrac{d}{dx}e^{x^2} = e^{x^2}(2x) = 2x\,e^{x^2}$

(b) $\dfrac{d}{dt}e^{-3t} = e^{-3t}(-3) = -3e^{-3t}$

(c) $\dfrac{d}{dx}\exp(\cos(\pi x)) = e^{\cos(\pi x)}(-\sin(\pi x)\,\pi)$

$= -\pi\sin(\pi x)\,e^{\cos(\pi x)}$

Since $\frac{d}{dx}e^x = e^x$, we have

$$\int e^u\,du = e^u + C.$$

Example: evaluate $\int xe^{x^2}\,dx$

Let $u = x^2$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$$= \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

Evaluate: $\int_0^1 e^{-2t}\,dt$

Let $u = -2t$

$du = -2\,dt$

$-\frac{1}{2}\,du = dt$

when $t=0,\ u=0$

$t=1,\ u=-2$

$$= \frac{-1}{2}\int_0^{-2} e^u\,du$$

$$= \frac{-1}{2} e^u \Big|_0^{-2} = \frac{-1}{2} e^{-2} - \left(\frac{-1}{2}\right) e^0$$

$$= \frac{1}{2} - \frac{1}{2e^2}$$

A General Result

Evaluate: $\displaystyle\int e^{ax}\,dx$ for a any nonzero constant.

Let $u = ax$ so that $\frac{1}{a}\,du = dx$

It follows that

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

Another General Result

Evaluate: $\displaystyle\int \frac{dx}{ax+b}$ for $a \neq 0$ and b any constants.

Let $u = ax+b$ so $\frac{1}{a}\,du = dx$

Then

$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| + C$$

Section 6.4*: General Logarithm and Exponential Functions

If $a > 0$ and $a \neq 1$, we define the **exponential function with base** a by

$$a^x = e^{x \ln a}.$$

Some Properties: Let x and y be real numbers, and a and b positive real numbers. Then

$$(1) \quad a^{x+y} = a^x a^y, \quad (2) \quad a^{x-y} = \frac{a^x}{a^y},$$

$$(3) \quad (a^x)^y = a^{xy}, \quad (4) \quad (ab)^x = a^x b^x$$

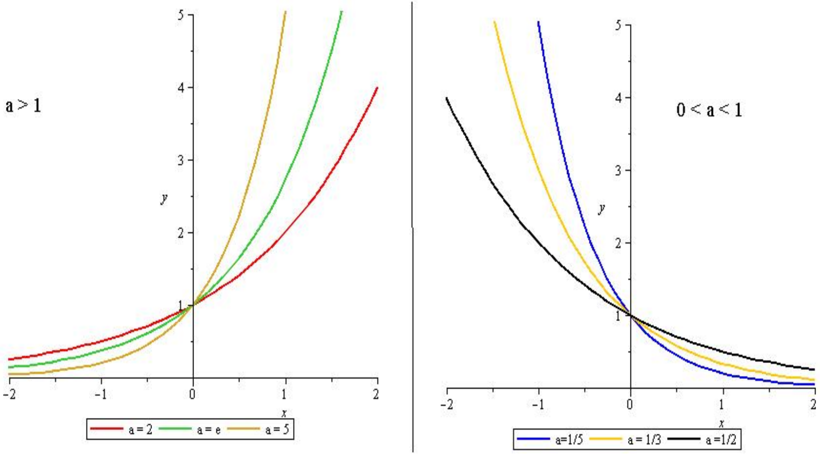

Figure: Plots of a^x for varying values of a. Note $a^0 = 1$ for any a.

Important Limits:

$$\lim_{x\to\infty} a^x = \begin{cases} \infty, & a > 1 \\ 0, & 0 < a < 1 \end{cases}, \qquad \lim_{x\to-\infty} a^x = \begin{cases} 0, & a > 1 \\ \infty, & 0 < a < 1 \end{cases}$$

For example, evaluate

$$\lim_{x\to\infty} \pi^x = \infty$$

$\pi > 1$

$$\lim_{t\to\infty} \left(\frac{3}{7}\right)^x = 0$$

$0 < \frac{3}{7} < 1$

Important Limits:

$$\lim_{x\to\infty} a^x = \begin{cases} \infty, & a > 1 \\ 0, & 0 < a < 1 \end{cases}, \qquad \lim_{x\to-\infty} a^x = \begin{cases} 0, & a > 1 \\ \infty, & 0 < a < 1 \end{cases}$$

For example, evaluate

$$\lim_{y\to 1^-} 2^{\frac{1}{y-1}} = 0$$

$\lim_{y\to 1^-} \frac{1}{y-1} = -\infty$ and $2 > 1$

$$\lim_{x\to\infty} \frac{3^{-x}}{4^{-x}} = \infty$$

$\frac{3^{-x}}{4^{-x}} = \left(\frac{4}{3}\right)^x$ and $\frac{4}{3} > 1$

Differentiation and Integration

Let $a > 0$ and $a \neq 1$.

$$\frac{d}{dx}a^x = a^x \ln a$$

$a^x = e^{x \ln a}$ and $x \ln a$ is a constant times x

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

Evaluate

(a) $\dfrac{d}{dt}7^{2t+1}$ $= 7^{2t+1}(\ln 7)2$

$= 2(\ln 7)\, 7^{2t+1}$

(b) $\int 3^x \, dx$

$$= \frac{3^x}{\ln 3} + C$$

(c) $\int_0^2 x\left(\frac{1}{2}\right)^{x^2}\,dx$

Let $u = x^2$, $du = 2x\,dx$

If $x=0$, $u=0$
$x=2$, $u=4$

$$= \frac{1}{2}\int_0^4 \left(\frac{1}{2}\right)^u du$$

$$= \frac{1}{2}\left.\frac{\left(\frac{1}{2}\right)^u}{\ln\left(\frac{1}{2}\right)}\right|_0^4 = \frac{1}{2}\frac{\left(\frac{1}{2}\right)^4}{(-\ln 2)} - \frac{1}{2}\frac{\left(\frac{1}{2}\right)^0}{(-\ln 2)}$$

$$= \frac{1}{2\ln 2} - \frac{1}{32\ln 2}$$

Exponential versus Power Functions

Don't confuse exponential functions (exponent is a variable) with power functions (base is a variable)!

Power: $f(x) = x^k$

Exponential: $f(x) = a^x$

$$\frac{d}{dx}x^2 = 2x, \quad \text{compare to} \quad \frac{d}{dx}2^x = 2^x \ln 2$$

$$\int x^3\,dx = \frac{x^4}{4} + C, \quad \text{compare to} \quad \int 3^x\,dx = \frac{3^x}{\ln 3} + C$$

Logarithmic Differentiation for hybrid functions

Use the natural logarithm to evaluate the derivative

$$\frac{d}{dx}x^x$$

$\ln y = \ln(x^x) = x\ln x$ hence

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$\boxed{\frac{dy}{dx} = x^x(\ln x + 1)}$$

General Logarithm

If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is one-to-one and has inverse function called the **logarithmic function with base** a**.** This inverse is denoted and defined by

$$\log_a x = y \quad \text{if and only if} \quad a^y = x.$$

Remark: $\log_e x = \ln x$

Change of Base Formula

We can translate between various bases via the formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

for any a, b, and x positive with $a \neq 1$ and $b \neq 1$.

In particular

$$\log_a x = \frac{\ln x}{\ln a}.$$

Derivative Formula

For $a > 0$ and $a \neq 1$, find a formula for

$$\frac{d}{dx} \log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$$

$$= \frac{1}{x \ln a}$$

Evaluate $\dfrac{d}{dx} \log_2 x = \dfrac{1}{x \ln 2}$

Evaluate

$$\frac{d}{dx}\log_6(x^3+3x^2) = \frac{3x^2+6x}{(x^3+3x^2)\ln 6}$$

e as a limit

Since $f(x) = \ln x$ satisfies $f'(1) = 1/1 = 1$, we have

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \ln(1+h) \\ &= \lim_{h\to 0} \ln\left(1+h\right)^{1/h} = 1 \end{aligned}$$

Since e^x is continuous, we can exponentiate. Writing x instead of h

$$e = e^1 = e^{\lim_{x\to 0} \ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0} (1+x)^{1/x}$$

Well known limits

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$