January 9 Math 2306 sec. 54 Spring 2019

Section 1: Concepts and Terminology

We have defined *differential equations*, and started to define certain characteristics and categories:

- Ordinary differential equations (ODEs) have one independent variable; partial differential equations (PDEs) have two or more independent variables.
- The **order** of an equation is equal to largest order of derivative appearing in the equation.
- An n^{th} order equation in *normal form* looks like $\dfrac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$ for some function f.

Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x, y, y').$$

Differential Form: A first order equation may appear in the form

$$\underbrace{M(x, y)\, dx + N(x, y)\, dy}_{\text{Differential form}} = 0$$

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Differential forms may be written in normal form in a couple of ways.

If $N(x,y) \neq 0$ for all relevant (x,y), we can write

$$N(x,y)\,dy = -M(x,y)\,dx$$

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

Similarly, if $M(x,y) \neq 0$, we could write

$$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)}$$

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Linear ODE

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The key characteristics here are:

- y and any of its derivatives can only appear as themselves (to the first power),
- coefficients of y and its derivatives may depend on the **independent** variable, but not on y or its derivatives,

Linear ODE

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x).$$

If we define the operation L by

$$Ly=a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y$$

then L is a **linear operator** in the sense that

$$L(Cy)=CLy\quad\text{for any constant } C,\quad\text{and}$$

$$L(y_1+y_2)=Ly_1+Ly_2,$$

for any pair of sufficiently differentiable functions y_1 and y_2.

Examples (Linear -vs- Nonlinear)

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The following are linear.

$$y'' + 4y = 0$$

Of the form

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

where

$a_2(x) = 1$

$a_1(x) = 0$

$a_0(x) = 4$

$g(x) = 0$

$$t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_2(t)x'' + a_1(t)x' + a_0(t)x = g(t)$

$a_2(t) = t^2$

$a_1(t) = 2t$

$a_0(t) = -1$

$g(t) = e^t$

Examples (Linear -vs- Nonlinear)

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The following are nonlinear.

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

$$y''' + \underbrace{(y')^3 y'}_{\text{nonlinear term}} = x^3$$

$$u'' + u' = \underbrace{\cos u}_{\text{nonlinear term}}$$

u is dependent, so

cos u is nonlinear term

For example

$$\cos(u_1+u_2) \neq \cos u_1 + \cos u_2$$

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$ $\Rightarrow$ $y''' + y'' + 2ty' - y = \cos t$

Independent variable t

dependent y

Order 3^{rd}

It is linear

(b) $\ddot{\theta}+\frac{g}{\ell}\sin\theta=0$ g and ℓ are constant

Order 2nd

Independent variable is time t

dependent is θ

θ is dependent, $\sin\theta$ is a nonlinear term

The ODE is nonlinear

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval[1] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty), \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

ϕ has 2 continuous derivatives on I (it has ∞-many).

Let's substitute $y = \phi(t)$ into the Equation

For $y = 3e^{2t}$

$y' = 6e^{2t}$

$y'' = 12e^{2t}$

$$y'' - y' - 2y =$$

$$12e^{2t} - 6e^{2t} - 2\left(3e^{2t}\right) =$$

$$12e^{2t} - 6e^{2t} - 6e^{2t} =$$

$$0 \cdot e^{2t} = 0$$

$$0 = 0$$

So $\phi(t) = 3e^{2t}$ is a solution.