# January 9 Math 2306 sec. 57 Spring 2018

## Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then $y$ is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even $dy/dx$ is differentiable with $d^2y/dx^2 = -4\cos(2x)$.

# Suppose $y = \cos(2x)$

Note that $$\frac{d^2y}{dx^2} + 4y = 0.$$

$$\frac{d^2y}{dx^2} = -4\cos(2x)$$

So

$$\frac{d^2y}{dx^2} + 4y = -4\cos(2x) + 4\cos(2x) = 0$$

# A *differential* equation

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a **differential equation.**

**Questions:** If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it? Also, is $\cos(2x)$ the only possible function that $y$ could be?

# Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

**Solving** a differential equation refers to determining the dependent variable(s)—as function(s).

**Independent Variable:** will appear as one that derivatives are taken with respect to.

**Dependent Variable:** will appear as one that derivatives are taken of.

# Independent and Dependent Variables

Often, the derivatives indicate which variable is which:

$$\frac{dy}{dx} \qquad \frac{du}{dt} \qquad \frac{dx}{dr}$$

$\leftarrow$ dependent

independent

$\leftarrow$ dependent

independent

$\Downarrow$

u is a function of t

# Classifications

**Type:** An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$\partial$ partial derivative symbol

[1] These are the subject of this course.

# Classifications

**Order:** The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$

1st derivative

1st order ODE

$$y''' + (y')^4 = x^3$$

3rd order ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

← highest derivative is 2nd

2nd order PDE

# Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if $s$ is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

# Notations and Symbols

An $n^{th}$ order ODE, with independent variable $x$ and dependent variable $y$ can always be expressed as an equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

where $F$ is some real valued function of $n + 2$ variables.

**Normal Form:** If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

2nd order looks like $F(x, y, y', y'') = 0$

when $F(x, y, y', y'') = y'' + 4y$

In normal form $\frac{d^2y}{dx^2} = f(x, y, y')$

where $f(x, y, y') = -4y$

# Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x,y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x,y,y').$$

**Differential Form:** A first order equation may appear in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Differential forms may be written in normal form in a couple of ways.

If $N(x,y) \neq 0$, we can rearrange

$$N(x,y)\,dy = -M(x,y)\,dx$$

$$dy = \frac{-M(x,y)}{N(x,y)}\,dx \Rightarrow \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

If $M(x,y) \neq 0$, we can also write this as

$$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)}$$

# Classifications

**Linearity:** An $n^{th}$ order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side $g$ may depend on the independent variable but not on the dependent variable or any of its derivatives.

$y, y', \ldots, y^{(n)}$ can only appear to the $1^{st}$ power,

$a_i$ can <u>only</u> depend on $x$

$g$ can only depend on $X$

# Examples (Linear -vs- Nonlinear)

$$y'' + 4y = 0$$

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$a_2(x) = 1$, $a_1(x) = 0$

$a_0(x) = 4$, $g(x) = 0$

$$t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_2(t)x'' + a_1(t)x' + a_0(t)x = g(t)$

$a_2(t) = t^2$, $a_1(t) = 2t$

$a_0(t) = -1$, $g(t) = e^t$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

$\frac{d^3y}{dx^3} + \underbrace{\left(\frac{dy}{dx}\right)^3}_{\text{non linear}} \frac{dy}{dx} = x^3$

$$u'' + u' = \cos u$$

non linear term

u is dependent it's inside the cosine function

# Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$

Order : $3^{rd}$

rewrite

Depend : $y$

Indepent : $t$

$y''' + y'' + 2ty' - y = \cos t$

Looks like

Linear

$a_3(t)\, y''' + a_2(t)\, y'' + a_1(t) y' + a_0(t)\, y = g(t)$

(b) $\ddot{\theta}+\dfrac{g}{\ell}\sin\theta=0$ $\quad g$ and $\ell$ are constant

non linear

since

$\theta$ is dependent

order : $2^{nd}$

depend : $\theta$

Independ : time $t$

Non linear

# Solution of $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

**Definition:** A function $\phi$ defined on an interval[2] $I$ and possessing at least $n$ continuous derivatives on $I$ is a **solution** of (*) on $I$ if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

**Definition:** An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[2]The interval is called the *domain of the solution* or the *interval of definition*.

# Examples:

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty), \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

$\phi$ has more than 2 continuous derivatives on $I$.

$\phi' = 6e^{2t}$ and $\phi'' = 12e^{2t}$. Set $y = \phi$

Then $y'' - y' - 2y = 12e^{2t} - 6e^{2t} - 2(3e^{2t})$

$= 12e^{2t} - 6e^{2t} - 6e^{2t}$

$= 12e^{2t} - 12e^{2t} = 0$

So $\phi$ solves the ODE.