January 9 Math 2306 sec. 60 Spring 2018

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$.

Suppose $y = \cos(2x)$

Note that $$\frac{d^2y}{dx^2} + 4y = 0.$$

$$\frac{d^2y}{dx^2} = -4\cos(2x) \quad \text{so}$$

$$\frac{d^2y}{dx^2} + 4y = -4\cos(2x) + 4\cos(2x) = 0$$

A *differential* equation

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a **differential equation.**

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it? Also, is $\cos(2x)$ the only possible function that y could be?

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

Independent and Dependent Variables

Often, the derivatives indicate which variable is which:

$$\frac{dy}{dx} \qquad \frac{du}{dt} \qquad \frac{dx}{dr}$$

depend

independ

dependent

independent

implies u is a function of t

i.e. $u = f(t)$

Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

∂ partial derivative symbol.

[1] These are the subject of this course.

Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

1st derivative

$$\frac{dy}{dx} - y^2 = 3x$$

1st order ODE

3rd 1st

$$y''' + (y')^4 = x^3$$

3rd order ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

1st 2nd

2nd order PDE

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

Has the form $F(x,y,y',y'') = 0$

where $F(x,y,y',y'') = y'' + 4y$

In normal form $\frac{d^2y}{dx^2} = f(x,y,y')$ where

$f(x,y,y') = -4y$.

Note $\frac{d^2y}{dx^2} + 4y = 0 \Rightarrow \frac{d^2y}{dx^2} = -4y$

Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x, y, y').$$

Differential Form: A first order equation may appear in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

Differential forms may be written in normal form in a couple of ways.

If $N(x,y) \neq 0$, then we can rearrange

$$N(x,y)\, dy = -M(x,y)\, dx \quad \Rightarrow \quad dy = \frac{-M(x,y)}{N(x,y)}\, dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

Alternatively, if $M(x,y) \neq 0$, then we can write

$$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)}$$

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

Note, no dependent variable can be squared or rooted, or in another function.

Examples (Linear -vs- Nonlinear)

both linear

$$y'' + 4y = 0 \qquad\qquad t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$a_2(x) = 1 \qquad a_1(x) = 0$

$a_0(x) = 4 \qquad g(x) = 0$

$a_2(t)x'' + a_1(t)x' + a_0(t)x = g(t)$

$a_2(t) = t^2, \quad a_1(t) = 2t$

$a_0(t) = -1, \quad g(t) = e^t$

Both nonlinear

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3 \qquad\qquad u'' + u' = \cos u$$

$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^3 \frac{dy}{dx} = x^3$

↑ coef. depends on y

u is dependent.

Cos(u) makes the equation nonlinear

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$

Order: 3rd order

Depend: y

Independ: t

$y''' + y'' + 2ty' - y = \cos t$

Looks like

$a_3(t)\, y''' + a_2(t)\, y'' + a_1(t)\, y' + a_0(t)\, y = g(t)$

It is

linear