In a certain city, ABC shipping has one receiving (A) and two distribution hubs (B & C). On a given day, 80 packages enter center A and will be distributed to hubs B and C for delivery. Twenty packages will go to a major client from hub C, the rest are to be distributed in quantities $x_1, \ldots, x_4$ among the hubs and out for delivery.
Motivating Example

Figure: Distribution Scheme
Equations for Package Quantities

Assuming all of the packages are delivered to customers outside of the shipping company, the quantities $x_1, \ldots, x_4$ have to satisfy the equations

\[
\begin{align*}
x_1 + x_3 &= 20 \\
x_2 - x_3 - x_4 &= 0 \\
x_1 + x_2 &= 80
\end{align*}
\]
Questions

- Is there a set of numbers $x_1, \ldots, x_4$ that satisfy all of the equations?

- If there is a set of numbers, is it the only one?

- If we could find numbers $x_1, \ldots, x_4$, and then the input 80 changed (say on another day), do we have to do all the work again? Or is there a way to generalize our finding?
We begin with a linear \((algebraic)\) equation in \(n\) variables \(x_1, x_2, \ldots, x_n\) for some positive integer \(n\).

A \textbf{linear equation} can be written in the form

\[
a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b.\]

The numbers \(a_1, \ldots, a_n\) are called the \textit{coefficients}. These numbers and the right side \(b\) are real (or complex) constants that are \textbf{known}.
Examples of Equations that are or are not Linear

\[ 2x_1 = 4x_2 - 3x_3 + 5 \quad \text{and} \quad 12 - \sqrt{3}(x + y) = 0 \]

\[ x_1 + 3x_3 = \frac{1}{x_2} \quad \text{and} \quad xyz = \sqrt{w} \]
A Linear System is a collection of linear equations in the same variables

\[\begin{align*}
2x_1 + x_2 - 3x_3 + x_4 &= -3 \\
-x_1 + 3x_2 + 4x_3 - 2x_4 &= 8
\end{align*}\]

\[\begin{align*}
x + 2y + 3z &= 4 \\
3x + 12z &= 0 \\
2x + 2y - 5z &= -6
\end{align*}\]
Some terms

- A **solution** is a list of numbers \((s_1, s_2, \ldots, s_n)\) that reduce each equation in the system to a true statement upon substitution.

- A **solutions set** is the set of all possible solutions of a linear system.

- Two systems are called **equivalent** if they have the same solution set.
An Example

\[ 2x - y = -1 \]
\[ -4x + 2y = 2 \]

(a) Show that \((1, 3)\) is a solution.
An Example Continued

\[
2x - y = -1 \\
-4x + 2y = 2
\]

(b) Note that \( \{(x, y) \mid y = 2x + 1\} \) is the solution set.
The Geometry of 2 Equations with 2 Variables

Figure: The system \( x - y = -1 \) and \( 2x + y = 3 \) with solution set \( \{(2/3, 5/2)\} \). These equations represent lines that intersect at one point.
The Geometry of 2 Equations with 2 Variables

Figure: The system $x - y = -1$ and $2x - 2y = -2$ with solution set \{$(x, y)|y = x + 1$\}. Both equations represent the same line which share all common points as solutions.
The Geometry of 2 Equations with 2 Variables

Figure: The system $x - y = -1$ and $2x - 2y = 2$ with solution set $\emptyset$. These equations represent parallel lines having no common points.
Theorem

A linear system of equations has exactly one of the following:

i  No solution, or
ii  Exactly one solution, or
iii Infinitely many solutions.

Terms: A system is **consistent** if it has at least one solution (cases ii and iii), and **inconsistent** if it has no solutions (case i).

Two critical questions about any linear system are: (1) Does it have a solution? (existence), and (2) If it has a solution, is there only one? (uniqueness)
Matrices

**Definition:** A matrix is a rectangular array of numbers. Its **size** (a.k.a. dimension/order) is $m \times n$ (read ”$m$ by $n$”) where $m$ is the number of rows and $n$ is the number of columns the matrix has.

Examples:

$$
\begin{bmatrix}
2 & 0 & -1 & 3 \\
1 & 1 & 13 & -4 \\
12 & -3 & 2 & -2
\end{bmatrix},
\begin{bmatrix}
2 & 0 \\
4 & 4 \\
3 & -5
\end{bmatrix}
$$
Linear System: Coefficient Matrix

Given any linear system of equations, we can associate two matrices with the system. These are the coefficient matrix and the augmented matrix¹.

Example:

\[
\begin{align*}
  x_1 & + 2x_2 - x_3 = -4 \\
  2x_1 & + x_3 = 7 \\
  x_1 & + x_2 + x_3 = 6 
\end{align*}
\]

¹Note that like variables should be lined up vertically!
Linear System: Augmented Matrix

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix.

Example:

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= -4 \\
2x_1 + x_3 &= 7 \\
x_1 + x_2 + x_3 &= 6
\end{align*}
\]
Legitimate Operations for Solving a System

We can perform three basic operations without changing the solution set of a system. These are

▶ swap the order of any two equations (swap),

▶ multiply an equation by any nonzero constant (scale), and

▶ replace an equation with the sum of itself and a nonzero multiple of any other equation (replace).
Some Operation Notation

Notation

- Swap equations $i$ and $j$:
  \[ E_i \leftrightarrow E_j \]

- Scale equation $i$ by $k$:
  \[ kE_i \rightarrow E_i \]

- Replace equation $j$ with the sum of itself and $k$ times equation $i$:
  \[ kE_i + E_j \rightarrow E_j \]
Solve the following system of equations by *elimination*. Keep tabs on the augmented matrix at each step.

\[
\begin{align*}
    x_1 & + 2x_2 & - x_3 &= -4 \\
    2x_1 & & + x_3 &=  7 \\
    x_1 & + x_2 & + x_3 &=  6
\end{align*}
\]
Elementary Row Operations

If any sequence of the following operations are performed on a matrix, the resulting matrix is **row equivalent**.

i. Replace a row with the sum of itself and a multiple of another row (**replacement**).

ii. Interchange any two rows (**row swap**).

iii. Multiply a row by any nonzero constant (**scaling**).

**Theorem:** If the augmented matrices of two linear systems are row equivalent, then the systems have the same solution set. (i.e. The systems are equivalent!)
A key here is *structure*!

Consider the following augmented matrix. Determine if the associated system is consistent or inconsistent. If it is consistent, determine the solution set.

\[
\begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & -2
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 1 & -1 & 4 \\
0 & 0 & 0 & 3 \\
\end{bmatrix}
\]
\[
\begin{pmatrix}
1 & 1 & -1 & 0 \\
0 & 1 & 1 & 4 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Section 1.2: Row Reduction and Echelon Forms

**Definition:** A matrix is in **echelon form** (a.k.a. **row echelon form**) if the following properties hold

i. Any row of all zeros are at the bottom.

ii. The first nonzero number (called the **leading entry**) in a row is to the right of the first nonzero number in all rows above it.

iii. All entries below a leading entry are zeros.

<table>
<thead>
<tr>
<th>Is</th>
<th>Is Not</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
2 & 1 & 3 \\
0 & -1 & 1 \\
0 & 0 & 7 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & -3 \\
0 & 1 & 0 \\
0 & 0 & 4 \\
\end{pmatrix}
\] |
Reduced Echelon Form

**Definition:** A matrix is in **reduced echelon form** (a.k.a. **reduced row echelon form**) if it is in echelon form and the following additional properties hold

1. The leading entry of each row is 1 (called a *leading 1*), and
2. Each leading 1 is the only nonzero entry in its column.

<table>
<thead>
<tr>
<th>Is</th>
<th>Is Not</th>
</tr>
</thead>
</table>
| \[
  \begin{bmatrix}
  1 & 1 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0 \\
  \end{bmatrix}
  \] | \[
  \begin{bmatrix}
  1 & 1 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0 \\
  \end{bmatrix}
  \] |
Example (finding ref’s and rref’s)

Find an echelon form for the following matrix using elementary row operations.

\[
\begin{bmatrix}
2 & 1 & 3 \\
4 & 3 & 6 \\
0 & 3 & 2 \\
\end{bmatrix}
\]
Find the reduced echelon form for the following matrix.

\[
\begin{bmatrix}
2 & 1 & 3 \\
4 & 3 & 6 \\
0 & 3 & 2
\end{bmatrix}
\]
Theorem: The reduced row echelon form of a matrix is unique.

This allows the following unambiguous definition:

Definition: A pivot position in a matrix $A$ is a location that corresponds to a leading 1 in the reduced echelon form of $A$. A pivot column is a column of $A$ that contains a pivot position.
Identify the pivot position and columns given...

\[ A \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \quad \text{rref of } A \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
Complete Row Reduction isn’t needed to find Pivots

Find the pivot positions and pivot columns of the matrix

\[
\begin{bmatrix}
1 & 1 & 4 \\
-2 & 1 & -2 \\
1 & 0 & 2
\end{bmatrix}
\]

This matrix has an ref and rref

\[
\begin{bmatrix}
1 & 1 & 4 \\
0 & 3 & 6 \\
0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix},
\]

respectively.
Row Reduction Algorithm
To obtain an echelon form, we work from left to right beginning with the top row working downward.

\[
\begin{bmatrix}
0 & 3 & -6 & 4 & 6 \\
3 & -7 & 8 & 8 & -5 \\
3 & -9 & 12 & 6 & -9
\end{bmatrix}
\]

\((R_1 \leftrightarrow R_3)\)

Step 1: The left most column is a pivot column. The top position is a pivot position.
Step 2: Get a nonzero entry in the top left position by row swapping if needed.
Row Reduction Algorithm

\[
\begin{array}{cccccc}
3 & -9 & 12 & 6 & -9 \\
3 & -7 & 8 & 8 & -5 \\
0 & 3 & -6 & 4 & 6 \\
\end{array}
\]

**Step 3:** Use row operations to get zeros in all entries below the pivot.
Row Reduction Algorithm

\[
\begin{bmatrix}
3 & -9 & 12 & 6 & -9 \\
0 & 2 & -4 & 2 & 4 \\
0 & 3 & -6 & 4 & 6 \\
\end{bmatrix}
\]

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3.
Row Reduction Algorithm
Row Reduction Algorithm

To obtain a reduced row echelon form:

Step 5: Starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.

\[
\begin{bmatrix}
3 & -9 & 12 & 6 & -9 \\
0 & 1 & -2 & 1 & 2 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
Row Reduction Algorithm
Row Reduction Algorithm
Echelon Form & Solving a System

Remark: The row operations used to get an rref correspond to an equivalent system!

Consider the reduced echelon matrix, and describe the solution set for the associated system of equations (the one who’d have this as its augmented matrix).

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & \frac{-2}{4} & 4 \\
0 & 0 & 0 & 1 & 0 & \frac{-9}{4} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$
Consistent versus Inconsistent Systems

Consider each rref. Determine if the underlying system (the one with this as its augmented matrix) is consistent or inconsistent.

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & -3
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
An Existence and Uniqueness Theorem

**Theorem:** A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & b
\end{bmatrix},
\]

for some nonzero \( b \).

If a linear system is consistent, then it has

(i) exactly one solution if there are no free variables, or

(ii) infinitely many solutions if there is at least one free variable.
Section 1.3: Vector Equations

**Definition:** A matrix that consists of one column is called a **column vector** or simply a **vector**.

The set of vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x_1$ and $x_2$ any real numbers is denoted by $\mathbb{R}^2$ (read ”R two”). It’s the set of all real ordered pairs.
Geometry

Each vector \[\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\] corresponds to a point in the Cartesian plane. We can equate them with ordered pairs written in the traditional format
\[\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2).\] This is not to be confused with a row matrix.

\[\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq [x_1 \ x_2]\]

We can identify vectors with points or with directed line segments emanating from the origin (little arrows).
Geometry

Figure: Vectors characterized as points, and vectors characterized as directed line segments.
Algebraic Operations

Let \( \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \), \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \), and \( c \) be a scalar.

Scalar Multiplication: The scalar multiple of \( \mathbf{u} \)

\[
    c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.
\]

Vector Addition: The sum of vectors \( \mathbf{u} \) and \( \mathbf{v} \)

\[
    \mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.
\]

Vector Equivalence: Equality of vectors is defined by

\[
    \mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.
\]

\(^2\)A scalar is an element of the set from which \( u_1 \) and \( u_2 \) come. For our purposes, a scalar is a real number.
Examples

\[ u = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad w = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \]

Evaluate

(a) \(-2u\)

(b) \(-2u + 3v\)

Is it true that \(w = -\frac{3}{4}u\)?
Figure: Left: $\frac{1}{2}(-4, 1) = (-2, 1/2)$. Right: $(-4, 1) + (2, 5) = (-2, 6)$
Geometry of Algebra with Vectors

**Scalar Multiplication:** stretches or compresses a vector but can only change direction by an angle of $0$ (if $c > 0$) or $\pi$ (if $c < 0$). We’ll see that $0u = (0, 0)$ for any vector $u$. 
**Geometry of Algebra with Vectors**

**Vector Addition:** The sum $\mathbf{u} + \mathbf{v}$ of two vectors (each different from $(0, 0)$) is the fourth vertex of a parallelogram whose other three vertices are $(u_1, u_2)$, $(v_1, v_2)$, and $(0, 0)$. 
Vectors in $\mathbb{R}^n$

A vector in $\mathbb{R}^3$ is a $3 \times 1$ column matrix. These are ordered triples. For example

$$a = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \text{or} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A vector in $\mathbb{R}^n$ for $n \geq 2$ is a $n \times 1$ column matrix. These are ordered $n$-tuples. For example

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The Zero Vector: is the vector whose entries are all zeros. It will be denoted by $\mathbf{0}$ or $\vec{0}$ and is not to be confused with the scalar 0.
Algebraic Properties on $\mathbb{R}^n$

For every $u, v, \text{ and } w$ in $\mathbb{R}^n$ and scalars $c$ and $d$\footnote{The term $-u$ denotes $(-1)u$.}:

\begin{align*}
(i) \quad u + v &= v + u \\
(ii) \quad (u + v) + w &= u + (v + w) \\
(iii) \quad u + 0 &= 0 + u = u \\
(iv) \quad u + (-u) &= -u + u = 0 \\
(v) \quad c(u + v) &= cu + cv \\
(vi) \quad (c + d)u &= cu + du \\
(vii) \quad c(du) &= d(cu) = (cd)u \\
(viii) \quad 1u &= u
\end{align*}
Definition: Linear Combination

A linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ is a vector $\mathbf{y}$ of the form

$$\mathbf{y} = c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p$$

where the scalars $c_1, \ldots, c_p$ are often called weights.

For example, suppose we have two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Some linear combinations include

$$3\mathbf{v}_1, \quad -2\mathbf{v}_1 + 4\mathbf{v}_2, \quad \frac{1}{3} \mathbf{v}_2 + \sqrt{2} \mathbf{v}_1, \quad \text{and} \quad 0 = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$
Example

Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, and $b = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$. Determine if $b$ can be written as a linear combination of $a_1$ and $a_2$. 
Some Convenient Notation

Letting \( a_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \ a_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}, \) and in general \( a_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}, \) for \( j = 1, \ldots, n, \) we can denote the \( m \times n \) matrix whose columns are these vectors by

\[
[ a_1 \ a_2 \ \cdots \ a_n ] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.
\]

Note that each vector \( a_j \) is a vector in \( \mathbb{R}^m. \)
Vector and Matrix Equations

The vector equation

\[ x_1a_1 + x_2a_2 + \cdots + x_na_n = b \]

has the same solution set as the linear system whose augmented matrix is

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & a_n & b \\
\end{bmatrix}.
\] (1)

In particular, \( b \) is a linear combination of the vectors \( a_1, \ldots, a_n \) if and only if the linear system whose augmented matrix is given in (1) is consistent.
Definition of **Span**

Let \( S = \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) be a set of vectors in \( \mathbb{R}^n \). The set of all linear combinations of \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) is denoted by

\[
\text{Span}\{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} = \text{Span}(S).
\]

It is called the **subset of** \( \mathbb{R}^n \) **spanned by** (a.k.a. **generated by**) the set \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \).

To say that a vector \( \mathbf{b} \) is in \( \text{Span}\{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) means that there exists a set of scalars \( c_1, \ldots, c_p \) such that \( \mathbf{b} \) can be written as

\[
c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p.
\]
If $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then $\mathbf{b} = c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p$. From the previous result, we know this is equivalent to saying that the vector equation

$$x_1 \mathbf{v}_1 + \cdots + x_p \mathbf{v}_p = \mathbf{b}$$

has a solution. This is in turn the same thing as saying the linear system with augmented matrix $[\mathbf{v}_1 \cdots \mathbf{v}_p \mathbf{b}]$ is consistent.
Examples

Let $a_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $a_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine if $b = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is in $\text{Span}\{a_1, a_2\}$. 
(b) Determine if \( \mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix} \) is in \( \text{Span}\{\mathbf{a}_1, \mathbf{a}_2\} \).
Another Example

Give a geometric description of the subset of $\mathbb{R}^2$ given by $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. 
Span\{\textbf{u}\} in $\mathbb{R}^3$

If $\textbf{u}$ is any nonzero vector in $\mathbb{R}^3$, then Span\{\textbf{u}\} is a line through the origin parallel to $\textbf{u}$.
Span\{\mathbf{u}, \mathbf{v}\} \text{ in } \mathbb{R}^3

If \mathbf{u} and \mathbf{v} are nonzero, and nonparallel vectors in \mathbb{R}^3, then Span\{\mathbf{u}, \mathbf{v}\} is a plane containing the origin parallel to both vectors.

**Figure:** The red and blue vectors are \mathbf{u} and \mathbf{v}. The plane is the collection of all possible linear combinations. (A purple representative is shown.)
Example

Let \( \mathbf{u} = (1, 1) \) and \( \mathbf{v} = (0, 2) \) in \( \mathbb{R}^2 \). Show that for every pair of real numbers \( a \) and \( b \), that \( (a, b) \) is in \( \text{Span}\{\mathbf{u}, \mathbf{v}\} \).
Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

**Definition** Let $A$ be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of $A$ and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

is the linear combination of the columns of $A$ whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$  

(Note that the result is a vector in $\mathbb{R}^m$!)
Example

Find the product $Ax$. Simplify to the extent possible.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$
Example
Find the product $Ax$. Simplify to the extent possible.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \quad x = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$
Example
Write the linear system as a vector equation and then as a matrix equation of the form $Ax = b$.

\[
\begin{align*}
2x_1 & - 3x_2 + x_3 = 2 \\
x_1 & + x_2 + = -1
\end{align*}
\]
Theorem

If \( A \) is the \( m \times n \) matrix whose columns are the vectors \( a_1, a_2, \ldots, a_n \), and \( b \) is in \( \mathbb{R}^m \), then the matrix equation

\[
Ax = b
\]

has the same solution set as the vector equation

\[
x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b
\]

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

\[
[a_1 \ a_2 \ \cdots \ a_n \ b].
\]
Corollary

The equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$.

In other words, the corresponding linear system is consistent if and only if $b$ is in $\text{Span}\{a_1, a_2, \ldots, a_n\}$.
Example
Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$
Theorem (first in a string of equivalency theorems)

Let $A$ be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $b$ in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.

(b) Each $b$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.

(c) The columns of $A$ span $\mathbb{R}^m$.

(d) $A$ has a pivot position in every row.

(Note that statement (d) is about the coefficient matrix $A$, not about an augmented matrix $[A \ b]$.)
Computing $Ax$

We can use a *row-vector* dot product rule. The $i^{th}$ entry is $Ax$ is the sum of products of corresponding entries from row $i$ of $A$ with those of $x$. For example

$$
\begin{bmatrix}
1 & 0 & -3 \\
-2 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix}
$$
\[
\begin{bmatrix}
2 & 4 \\
-1 & 1 \\
0 & 3 \\
\end{bmatrix}
\begin{bmatrix}
-3 \\
2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\]
Identity Matrix

We’ll call an $n \times n$ matrix with 1’s on the diagonal and 0’s everywhere else—i.e. one that looks like

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix}
$$

the $n \times n$ identity matrix and denote it by $I_n$. (We’ll drop the subscript if it’s obvious from the context.)

This matrix has the property that for each $x$ in $\mathbb{R}^n$

$$I_n x = x.$$
Theorem: Properties of the Matrix Product

If $A$ is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and $c$ is any scalar, then

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$. 
Section 1.5: Solution Sets of Linear Systems

**Definition** A linear system is said to be **homogeneous** if it can be written in the form

\[ Ax = 0 \]

for some \( m \times n \) matrix \( A \) and where \( 0 \) is the zero vector in \( \mathbb{R}^m \).

**Theorem:** A homogeneous system \( Ax = 0 \) always has at least one solution \( x = 0 \).

The solution \( x = 0 \) is called the **trivial solution**. A more interesting question for a homogeneous system is

**Does it have a nontrivial solution?**
Theorem

The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a) $2x_1 + x_2 = 0$
   $x_1 - 3x_2 = 0$
(b) \[
3x_1 + 5x_2 - 4x_3 = 0 \\
-3x_1 - 2x_2 + 4x_3 = 0 \\
6x_1 + x_2 - 8x_3 = 0
\]
(c) \[ x_1 - 2x_2 + 5x_3 = 0 \]
Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form \( \mathbf{x} = x_3 \mathbf{v} \). Example (c)’s solution set consisted of vectors that look like \( \mathbf{x} = x_2 \mathbf{u} + x_3 \mathbf{v} \). Since these are linear combinations, we could write the solution sets like

\[
\text{Span}\{\mathbf{u}\} \quad \text{or} \quad \text{Span}\{\mathbf{u}, \mathbf{v}\}.
\]

Instead of using the variables \( x_2 \) and/or \( x_3 \) we often substitute parameters such as \( s \) or \( t \).

The forms

\[
\mathbf{x} = s \mathbf{u}, \quad \text{or} \quad \mathbf{x} = s \mathbf{u} + t \mathbf{v}
\]

are called parametric vector forms.
Example

The parametric vector form of the solution set of
\[ x_1 - 2x_2 + 5x_3 = 0 \]
is
\[
\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.
\]

Question: What geometric object is that solution set?
Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

\[ 3x_1 + 5x_2 - 4x_3 = 7 \]
\[ -3x_1 - 2x_2 + 4x_3 = -1 \]
\[ 6x_1 + x_2 - 8x_3 = -4 \]
Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

\[ x = p + tv \]

with \( p \) and \( v \) fixed vectors and \( t \) a varying parameter. Also note that the \( tv \) part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

\( p \) is called a **particular solution**, and \( tv \) is called a solution to the associated homogeneous equation.
Theorem

Suppose the equation $Ax = b$ is consistent for a given $b$. Let $p$ be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the form

$$x = p + v_h,$$

where $v_h$ is any solution of the associated homogeneous equation $Ax = 0$.

We can use a row reduction technique to get all parts of the solution in one process.
Example

Find the solution set of the following system. Express the solution set in parametric vector form.

\[
\begin{align*}
    x_1 + x_2 - 2x_3 + 4x_4 &= 1 \\
    2x_1 + 3x_2 - 6x_3 + 12x_4 &= 4
\end{align*}
\]