January 9 Math 3260 sec. 55 Spring 2018

In a certain city, ABC shipping has one receiving (A) and two distribution hubs (B & C). On a given day, 80 packages enter center A and will be distributed to hubs B and C for delivery. Twenty packages will go to a major client from hub C, the rest are to be distributed in quantities $x_1, \ldots, x_4$ among the hubs and out for delivery.

Motivating Example

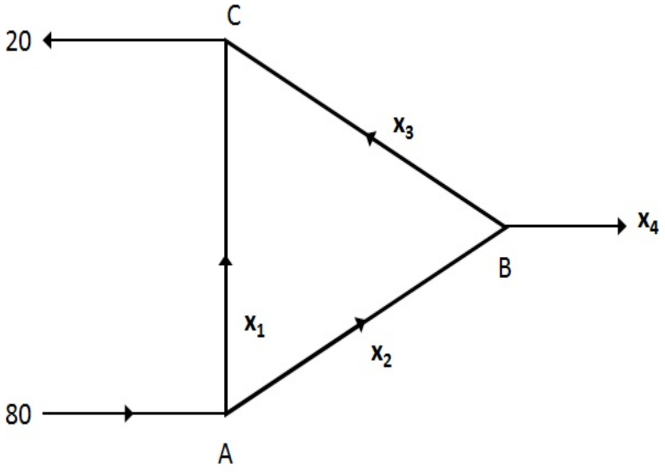

Figure: Distribution Scheme

Equations for Package Quantities

Assuming all of the packages are delivered to customers outside of the shipping company, the quantities $x_1, \ldots, x_4$ have to satisfy the equations

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & & & = & 20 \\ & & x_2 & - & x_3 & - & x_4 & = & 0 \\ x_1 & + & x_2 & & & & & = & 80 \end{array}$$

Questions

- Is there a set of numbers $x_1, \ldots, x_4$ that satisfy all of the equations?
- If there is a set of numbers, is it the only one?
- If we could find numbers $x_1, \ldots, x_4$, and then the input 80 changed (say on another day), do we have to do all the work again? Or is there a way to generalize our finding?

Section 1.1: Systems of Linear Equations

We begin with a linear (*algebraic*) equation in n variables x_1, x_2, ..., x_n for some positive integer n.

A **linear equation** can be written in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b.$$

The numbers $a_1, \ldots, a_n$ are called the *coefficients*. These numbers and the right side b are real (or complex) constants that are **known**.

Examples of Equations that are or are not Linear

both linear

$$2x_1 = 4x_2 - 3x_3 + 5 \quad \text{and} \quad 12 - \sqrt{3}(x + y) = 0$$

$2x_1 - 4x_2 + 3x_3 = 5$ $\qquad$ $\sqrt{3}\,x + \sqrt{3}\,y = 12$

both nonlinear

$$x_1 + 3x_3 = \frac{1}{x_2} \quad \text{and} \quad xyz = \sqrt{w}$$

nonlinear term

nonlinear product of variables

square root of variable

A *Linear System* is a collection of linear equations in the same variables

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array}$$

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4 \\ 3x & & & + & 12z & = & 0 \\ 2x & + & 2y & - & 5z & = & -6 \end{array}$$

Some terms

- A **solution** is a list of numbers $(s_1, s_2, \ldots, s_n)$ that reduce each equation in the system to a true statement upon substitution.

- A **solutions set** is the set of all possible solutions of a linear system.

- Two systems are called **equivalent** if they have the same solution set.

An Example

$$\begin{array}{rcrcr} 2x & - & y & = & -1 \\ -4x & + & 2y & = & 2 \end{array}$$

(a) Show that $(1, 3)$ is a solution.

(1, 3) means x = 1, y = 3.

1st eqn

$2(1) - (3) = 2 - 3 = -1$

The eqn is $-1 = -1$ an identity

2nd eqn

$-4(1) + 2(3) = -4 + 6 = 2$

The eqn is $2 = 2$ an identity.

So (1, 3) is a solution.

An Example Continued

$$\begin{array}{rcrcr} 2x & - & y & = & -1 \\ -4x & + & 2y & = & 2 \end{array}$$

(b) Note that $\{(x,y)|y = 2x+1\}$ is the solution set.

The 1^{st} eqn can be rearranged

$2x - y = -1 \Rightarrow -y = -2x - 1$

$\Rightarrow y = 2x + 1$ the set condition

The 2^{nd} eqn can be rearranged

$-4x + 2y = 2 \Rightarrow 2y = 4x + 2$

$y = 2x + 1$ also the set condition

The Geometry of 2 Equations with 2 Variables

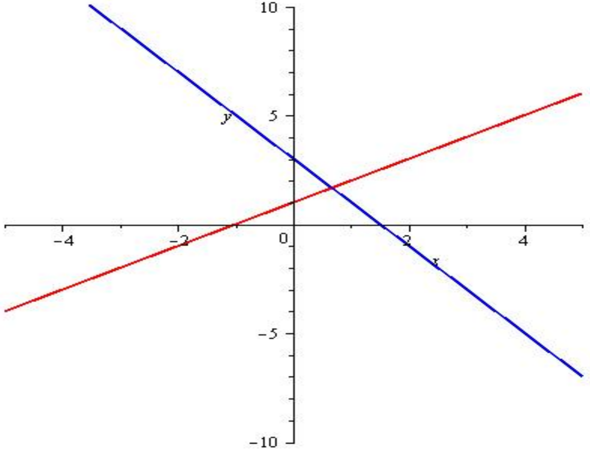

Figure: The system $x - y = -1$ and $2x + y = 3$ with solution set $\{(2/3, 5/2)\}$. These equations represent lines that intersect at one point.

The Geometry of 2 Equations with 2 Variables

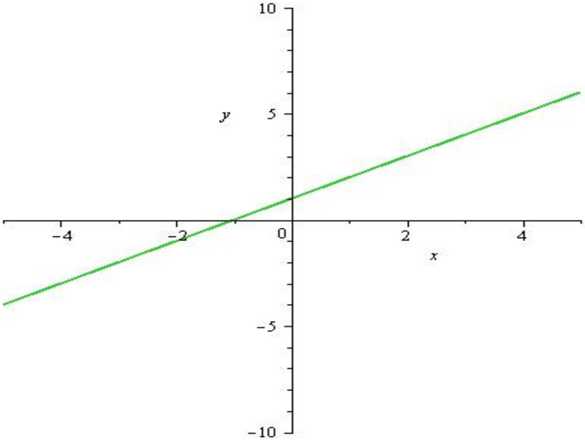

Figure: The system $x - y = -1$ and $2x - 2y = -2$ with solution set $\{(x, y) | y = x + 1\}$. Both equations represent the same line which share all common points as solutions.

The Geometry of 2 Equations with 2 Variables

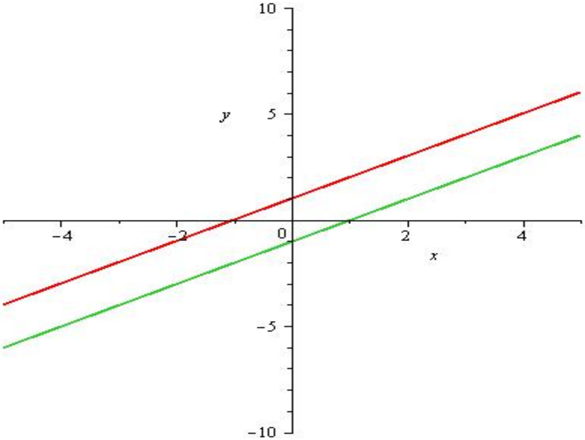

Figure: The system $x - y = -1$ and $2x - 2y = 2$ with solution set $\emptyset$. These equations represent parallel lines having no common points.

Theorem

A linear system of equations has exactly one of the following:

i No solution, or

ii Exactly one solution, or

iii Infinitely many solutions.

Terms: A system is
consistent if it has at least one solution (cases ii and iii), and
inconsistent if is has no solutions (case i).

Two critical questions about any linear system are: (1) Does it have a solution? (existence), and (2) If it has a solution, is there only one? (uniqueness)

Matrices

Definition: A matrix is a rectangular array of numbers. It's **size** (a.k.a. dimension/order) is $m \times n$ (read "m by n") where m is the number of rows and n is the number of columns the matrix has.

Examples:

4 columns   2 columns

3 rows $\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 1 & 13 & -4 \\ 12 & -3 & 2 & -2 \end{bmatrix}$,   3 rows $\begin{bmatrix} 2 & 0 \\ 4 & 4 \\ 3 & -5 \end{bmatrix}$

3×4 3×2

Linear System: Coefficient Matrix

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix[1].

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Coef. matrix has m rows where m = # equations
n columns where n = # variables

It contains the coefficients

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

[1] Note that like variables should be lined up vertically!

Linear System: Augmented Matrix

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix.

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Augmented has m rows m = # of eqns

n+1 columns n = # variables

Extra column for the right side

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Legitimate Operations for Solving a System

We can perform three basic operation **without** changing the solution set of a system. These are

- swap the order of any two equations **(swap)**,
- multiply an equation by any nonzero constant **(scale)**, and
- replace an equation with the sum of itself and a nonzero multiple of any other equation **(replace)**.

Some Operation Notation

Notation

- Swap equations i and j:
$$Ei \leftrightarrow Ej$$
- Scale equation i by k:
$$kEi \to Ei$$
- Replace equation j with the sum of itself and k times equation i:
$$kEi + Ej \to Ej$$

Solve the following system of equations by *elimination*. Keep tabs on the augmented matrix at each step.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Eliminate x_1

$-2E_1 + E_2 \to E_2$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & -4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$$\begin{array}{rcrcrcr} -2x_1 & - & 4x_2 & + & 2x_3 & = & 8 \\ 2x_1 & & & + & x_3 & = & 7 \end{array}$$

$-E_1 + E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -4x_2 + 3x_3 &= 15 \\ -x_2 + 2x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 0 & -1 & 2 & 10 \end{bmatrix}$$

$E_2 \leftrightarrow E_3$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & -4 & 3 & 15 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ -4x_2 + 3x_3 &= 15 \end{aligned}$$

$-4E_2 + E_3 \to E_3$

$$x_1 + 2x_2 - x_3 = -4$$
$$-x_2 + 2x_3 = 10$$
$$-5x_3 = -25$$

$$4x_2 - 8x_3 = -40$$
$$-4x_2 + 3x_3 = 15$$

$$-E_2 \to E_2$$
$$-\frac{1}{5}E_3 \to E_3$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & 0 & -5 & -25 \end{bmatrix}$$

$$x_1 + 2x_2 - x_3 = -4$$
$$x_2 - 2x_3 = -10$$
$$x_3 = 5$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

With substitution

$$x_3 = 5, \quad x_2 = -10 + 2x_3 = -10 + 10 = 0$$

$$x_1 = -4 - 2x_2 + x_3 = -4 + 0 + 5 = 1$$

The solution can be written as

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= 5 \end{aligned} \quad \text{or} \quad (1, 0, 5).$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

We ended with this augmented matrix. Note the triangular structure of the nonzero entries. The structure makes it easy to obtain a solution