In a certain city, ABC shipping has one receiving (A) and two distribution hubs (B & C). On a given day, 80 packages enter center A and will be distributed to hubs B and C for delivery. Twenty packages will go to a major client from hub C, the rest are to be distributed in quantities $x_1, \ldots, x_4$ among the hubs and out for delivery.
Motivating Example

Figure: Distribution Scheme
Equations for Package Quantities

Assuming all of the packages are delivered to customers outside of the shipping company, the quantities \( x_1, \ldots, x_4 \) have to satisfy the equations

\[
\begin{align*}
    x_1 + x_3 &= 20 \\
    x_2 - x_3 - x_4 &= 0 \\
    x_1 + x_2 &= 80
\end{align*}
\]
Questions

- Is there a set of numbers $x_1, \ldots, x_4$ that satisfy all of the equations?

- If there is a set of numbers, is it the only one?

- If we could find numbers $x_1, \ldots, x_4$, and then the input 80 changed (say on another day), do we have to do all the work again? Or is there a way to generalize our finding?
Section 1.1: Systems of Linear Equations

We begin with a linear \((algebraic)\) equation in \(n\) variables \(x_1, x_2, \ldots, x_n\) for some positive integer \(n\).

A linear equation can be written in the form

\[ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b. \]

The numbers \(a_1, \ldots, a_n\) are called the coefficients. These numbers and the right side \(b\) are real (or complex) constants that are known.
Examples of Equations that are or are not Linear

\[ 2x_1 = 4x_2 - 3x_3 + 5 \quad \text{and} \quad 12 - \sqrt{3}(x + y) = 0 \]

Both linear

\[ 2x_1 - 4x_2 + 3x_3 = 5 \]

\[ \sqrt{3}x + \sqrt{3}y = 12 \]

Both non-linear

\[ x_1 + 3x_3 = \frac{1}{x_2} \quad \text{and} \quad xyz = \sqrt{w} \]

\[ \text{reciprocal of variable} \]

\[ \text{product of variables} \]

\[ \text{square root of variable} \]
A Linear System is a collection of linear equations in the same variables

\begin{align*}
2x_1 & + x_2 - 3x_3 + x_4 = -3 \\
-x_1 + 3x_2 + 4x_3 - 2x_4 &= 8 \\
\end{align*}

\begin{align*}
x & + 2y + 3z = 4 \\
3x & + 12z = 0 \\
2x & + 2y - 5z = -6 \\
\end{align*}
Some terms

- A **solution** is a list of numbers \((s_1, s_2, \ldots, s_n)\) that reduce each equation in the system to a true statement upon substitution.

- A **solutions set** is the set of all possible solutions of a linear system.

- Two systems are called **equivalent** if they have the same solution set.
An Example

\[ 2x - y = -1 \]
\[ -4x + 2y = 2 \]

(a) Show that \((1, 3)\) is a solution.

\((1, 3)\) means \(x = 1\) and \(y = 3\)

1st eqn:
\[ 2(1) - 1(3) = 2 - 3 = -1 \]

The eqn is \(-1 = -1\) an identity

2nd eqn:
\[ -4(1) + 2(3) = -4 + 6 = 2 \]

The eqn is \(2 = 2\) an identity

So \((1, 3)\) is a solution.
An Example Continued

\[
2x - y = -1 \\
-4x + 2y = 2
\]

(b) Note that \( \{(x, y) | y = 2x + 1\} \) is the solution set.

The 1st eqn can be rearranged as:

\[
2x - y = -1 \Rightarrow -y = -2x - 1 \\
\Rightarrow y = 2x + 1
\]

Also, the 2nd equation gives:

\[
-4x + 2y = 2 \Rightarrow 2y = 4x + 2 \\
y = 2x + 1
\]
The Geometry of 2 Equations with 2 Variables

Figure: The system $x - y = -1$ and $2x + y = 3$ with solution set $\{(2/3, 5/2)\}$. These equations represent lines that intersect at one point.
The Geometry of 2 Equations with 2 Variables

Figure: The system $x - y = -1$ and $2x - 2y = -2$ with solution set $\{(x, y) | y = x + 1\}$. Both equations represent the same line which share all common points as solutions.
The Geometry of 2 Equations with 2 Variables

Figure: The system \( x - y = -1 \) and \( 2x - 2y = 2 \) with solution set \( \emptyset \). These equations represent parallel lines having no common points.
Theorem

A linear system of equations has exactly one of the following:

i. No solution, or 

ii. Exactly one solution, or

iii. Infinitely many solutions.

Terms: A system is **consistent** if it has at least one solution (cases ii and iii), and **inconsistent** if it has no solutions (case i).

Two critical questions about any linear system are: (1) Does it have a solution? (existence), and (2) If it has a solution, is there only one? (uniqueness)
Matrices

**Definition:** A matrix is a rectangular array of numbers. It’s **size** (a.k.a. dimension/order) is \( m \times n \) (read "\( m \) by \( n \)") where \( m \) is the number of rows and \( n \) is the number of columns the matrix has.

Examples:

\[
\begin{bmatrix}
2 & 0 & -1 & 3 \\
1 & 1 & 13 & -4 \\
12 & -3 & 2 & -2 \\
\end{bmatrix}, \quad \begin{bmatrix}
2 & 0 \\
4 & 4 \\
3 & -5 \\
\end{bmatrix}
\]

3×4  2×3
Linear System: Coefficient Matrix

Given any linear system of equations, we can associate two matrices with the system. These are the coefficient matrix and the augmented matrix\(^1\).

Example:

\[
\begin{align*}
2x_1 + x_3 &= 7 \\
2x_1 + x_3 &= 7 \\
x_1 + x_2 + x_3 &= 6
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 & -1 \\
2 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

\(^1\)Note that like variables should be lined up vertically!
Linear System: Augmented Matrix

Given any linear system of equations, we can associate two matrices with the system. These are the coefficient matrix and the augmented matrix.

Example:

\[
\begin{align*}
    x_1 + 2x_2 - x_3 &= -4 \\ 
    2x_1 + x_3 &= 7 \\ 
    x_1 + x_2 + x_3 &= 6 
\end{align*}
\]

An extra right column holds the right hand side.

Augmented is \( m \times (n+1) \) for the same \( m, n \) as before.

\[
\begin{bmatrix}
    1 & 2 & -1 & -4 \\
    2 & 0 & 1 & 7 \\
    1 & 1 & 1 & 6 
\end{bmatrix}
\]
Legitimate Operations for Solving a System

We can perform three basic operations without changing the solution set of a system. These are

- swap the order of any two equations (swap),

- multiply an equation by any nonzero constant (scale), and

- replace an equation with the sum of itself and a nonzero multiple of any other equation (replace).
Some Operation Notation

Notation

- Swap equations $i$ and $j$: $E_i \leftrightarrow E_j$
- Scale equation $i$ by $k$: $kE_i \rightarrow E_i$
- Replace equation $j$ with the sum of itself and $k$ times equation $i$: $kE_i + E_j \rightarrow E_j$