July 10 Math 1190 sec. 51 Summer 2017

Section 3.4: Newton's Method

Consider a function f that is differentiable on an interval (a, b). If there exists a root, some number α such that

$$f(\alpha) = 0,$$

in this interval, then Newton's Method provides an iterative scheme that may be able to find this number (at least with some degree of accuracy).

Iterative Scheme for Newton's Method

We start with a *guess* x_0. Then set

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, we can find a tangent to the graph of f at $(x_1, f(x_1))$ and update again

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Newton's Iteration Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \ldots$$

The sequence begins with a starting *guess* x_0 expected to be near the desired root.

Example: Equilibrium Population

If a population of animals (e.g. rabbits) changes at a rate jointly proportional to

- the current population, and
- the difference between the current population and the capacity of the environment

then that population (or more accurately population density) satisfies the logistic differential equation

rate of change of population

$$\frac{dx}{dt} = kx(M - x).$$

x–population density, k–a (scaled) net birth rate, M–a limit on the population that can be supported by the environment.

Example: Equilibrium Population

Suppose that individuals emigrate at a rate $E(x)$ where

$$E(x) = axe^{-\frac{x}{M}} \quad \text{(a is constant)}, \quad x \geq 0.$$

$* \ \frac{-x}{M} = \frac{-1}{M}x$

Show that the emigration rate is maximum when $x = M$.

We can find the critical number(s) of E.

$$E'(x) = a \cdot 1 \cdot e^{\frac{-x}{M}} + axe^{\frac{-x}{M}} \cdot \left(\frac{-1}{M}\right)$$

$$= ae^{\frac{-x}{M}}\left(1 - \frac{1}{M}x\right)$$

$E'(x)$ is never undefined.

$$E'(x) = 0 \Rightarrow \qquad a e^{\frac{-x}{M}}\left(1 - \frac{1}{M}x\right) = 0$$

$$e^{\frac{-x}{M}} > 0 \qquad 1 - \frac{1}{M}x = 0 \Rightarrow \quad x = M$$

We want to verify that E is maximum when $x = M$. We'll use the 2nd derivative test:

$$E''(x) = a e^{\frac{-x}{M}}\left(\frac{-1}{M}\right) - a\left(\frac{1}{M}\right)\left(1 \cdot e^{\frac{-x}{M}} + x e^{\frac{-x}{M}} \cdot \left(\frac{-1}{M}\right)\right)$$

$$= -a\frac{1}{M} e^{\frac{-x}{M}} - a\frac{1}{M} e^{\frac{-x}{M}} + a\left(\frac{1}{M}\right)^2 x e^{\frac{-x}{M}}$$

$$E''(x) = -2a\left(\frac{1}{M}\right) e^{\frac{-x}{M}} + a\left(\frac{1}{M}\right)^2 x e^{\frac{-x}{M}}$$

$$E''(x) = \frac{a}{M} e^{\frac{-x}{M}} \left(-2 + \frac{1}{M} x \right)$$

$$E''(M) = \frac{a}{M} e^{\frac{-M}{M}} \left(-2 + \frac{1}{M} \cdot M \right) = -\frac{a}{M} e^{-1} < 0$$

So E takes a maximum at the critical number M by the 2nd derivative test.

Example: Equilibrium Population

With emigration, the new model for the population is

$$\frac{dx}{dt} = kx(M - x) - axe^{-\frac{x}{M}}.$$

An **Equilibrium** population is one that doesn't change in time.

Find a function $f(x)$ whose root would be an equilibrium population.

If x doesn't change, then $\frac{dx}{dt} = 0$.

We want $0 = kx(M-x) - axe^{\frac{-x}{M}}$ so take

$$f(x) = kx(M-x) - axe^{\frac{-x}{M}}$$

Example: Equilibrium Population

Identify the Newton's Method formula that can be used to find the equilibrium population if $k = M = 1$ and $a = \frac{1}{2}$.

for $k = 1$, $M = 1$, $a = \frac{1}{2}$

$$f(x) = x(1-x) - \frac{1}{2} x e^{-x}$$

$$f'(x) = 1 - x - x - \frac{1}{2} e^{-x} - \frac{1}{2} x e^{-x} \cdot (-1)$$

$$= 1 - 2x - \frac{1}{2} e^{-x} + \frac{1}{2} x e^{-x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\boxed{x_{n+1} = x_n - \frac{x_n(1-x_n) - \frac{1}{2}x_n e^{-x_n}}{1 - 2x_n - \frac{1}{2}e^{-x_n} + \frac{1}{2}x_n e^{-x_n}}}$$

Estimating Square Roots

Suppose we wish to approximate the square root of a positive number a. Newton's Method provides a scheme for doing this.

Assuming that $\sqrt{a}$ is **NOT** already known, define a simple function $f(x)$ whose positive root would be the true value of $\sqrt{a}$.

The simplest f is

$$f(x) = x^2 - a$$

Estimating Square Roots

Write the Newton's Method scheme for finding $\sqrt{a}$.

$$f(x) = x^2 - a \quad , \quad f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n}$$

$$x_{n+1} = x_n - \left(\frac{x_n^2}{2x_n} - \frac{a}{2x_n}\right) = x_n - \frac{1}{2}x_n + \frac{a}{2x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}x_n + \frac{a}{2x_n}}$$

Estimating Square Roots

Using an initial guess of $x_0 = 1$, compute the iterates x_1 and x_2 that approximate $\sqrt{2}$.

$$x_{n+1} = \frac{1}{2}x_n + \frac{a}{2x_n} \quad \text{for} \quad a = 2$$

$$x_{n+1} = \frac{1}{2}x_n + \frac{2}{2x_n}$$

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$$

$$x_1 = \frac{1}{2}x_0 + \frac{1}{x_0}$$

$$= \frac{1}{2}(1) + \frac{1}{1} = \frac{1}{2} + 1 = \frac{3}{2}$$

$$x_2 = \frac{1}{2}x_1 + \frac{1}{x_1}$$

$$= \frac{1}{2}\left(\frac{3}{2}\right) + \frac{1}{\frac{3}{2}}$$

$$= \frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12}$$

$x_1 = \frac{3}{2}$ and $x_2 = \frac{17}{12}$

Square Root of 2

With $x_0 = 1$, the scheme

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$$

gives

$$\begin{aligned} x_1 &= \frac{3}{2} = 1.50000 \\ x_2 &= \frac{17}{12} = 1.41667 \\ x_3 &= \frac{577}{408} = 1.41422 \end{aligned}$$

The built in square root function on the TI-89 gives

$$\sqrt{2} = 1.41421, \quad \text{and} \quad \left| \sqrt{2} - \frac{577}{408} \right| = 2 \cdot 10^{-6}$$

Question

Suppose a decimal approximation to the number $\sqrt[3]{7}$ is needed (meaning that $\sqrt[3]{7}$ is NOT already known).

Which function f and initial iterate x_0 could be used with Newton's method to find such an approximation?

(a) $f(x) = x - \sqrt[3]{7}, \quad x_0 = 7$

(b) $f(x) = x^3 - 7, \quad x_0 = \sqrt[3]{7}$

$\sqrt[3]{7}$ isn't known, so those options don't make sense

(c) $f(x) = x^3 - 7, \quad x_0 = 2$

(d) $f(x) = x^2 - 7, \quad x_0 = 1$

Section 4.1: Related Rates

Motivating Example: A spherical balloon is being filled with air. Suppose that we know that the radius is increasing in time at a constant rate of 2 mm/sec. Can we determine the rate at which the surface area of the balloon is increasing at the moment that the radius is 10 cm?

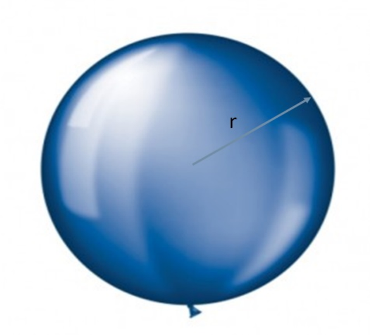


Figure: Spherical Balloon

Surface area changes as radius changes

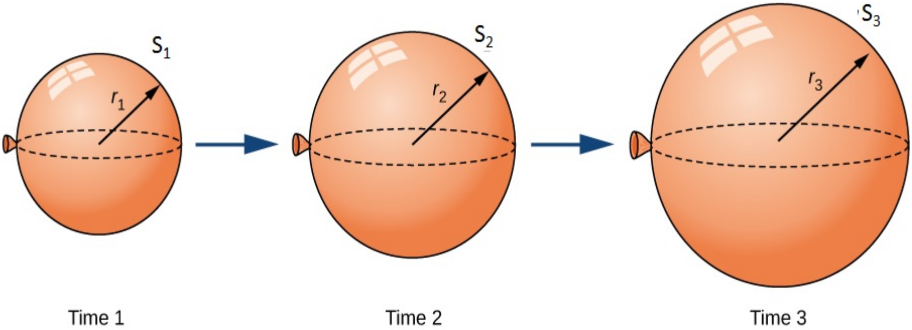


Figure: As a balloon inflates, the radius, surface area, and volume all change with time. Time is independent. Surface area depends on radius which in turn depends on time. $S = S(r) = S(r(t))$

Example Continued...

Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of time. Write an equation that relates

$$\frac{dS}{dt} \quad \text{to} \quad \frac{dr}{dt}.$$

rate of change of surface area

rate of change of radius.

By the chain rule

$$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$$

From $S = 4\pi r^2$, $\frac{dS}{dr} = 4\pi(2r) = 8\pi r$

$$\Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

Back to Our Balloon

Given this result, find the rate at which the surface area is changing when the radius is 10 cm.

The increasing rate of radius of 2 mm/sec is rate of change of r – i.e. $\frac{dr}{dt}$.

We have $\frac{dr}{dt} = 2 \frac{mm}{sec}$ and we're interested in the moment when $r = 10\,cm = 100\,mm$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt} \quad \text{when} \quad r = 100\,mm$$

$$\frac{dS}{dt} = 8\pi (100\,mm) \cdot 2 \frac{mm}{sec} = 1600\pi \frac{mm^2}{sec}$$

S is increasing at a rate of 1600π mm^2 per sec.

Example

A right circular cone of height h and base radius r has volume

$$V = \frac{\pi}{3} r^2 h.$$

(a) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ if r is constant.

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} \quad , \quad \frac{dV}{dh} = \frac{\pi}{3} r^2 \cdot 1$$

$$\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt}$$

Example Continued...

$$V = \frac{\pi}{3} r^2 h$$

Question (b) Find $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$ if h is constant.

(a) $\frac{dV}{dt} = \frac{2\pi}{3} r \frac{dr}{dt}$

(b) $\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dr}{dt}$

(c) $\frac{dV}{dt} = \frac{\pi}{3} r^2 h \frac{dr}{dt}$

(d) $\frac{dV}{dt} = \frac{2\pi}{3} rh \frac{dr}{dt}$

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt}$$

$$\frac{dV}{dr} = \frac{\pi}{3} h \, (2r)$$

And Continued Further...

$$V = \frac{\pi}{3} r^2 h$$

(c) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ and $\frac{dr}{dt}$ assuming neither r nor h is constant.

We need the product rule on $r^2 h$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} + \frac{dV}{dh}\frac{dh}{dt}$$

$$\frac{d}{dt}V = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}(2r)\frac{dr}{dt}h + \frac{\pi}{3}r^2 \cdot 1 \frac{dh}{dt} = \frac{2\pi}{3} rh\frac{dr}{dt} + \frac{\pi}{3}r^2\frac{dh}{dt}$$

Example

Suppose x and y are differentiable functions of time and that

$$z = xy - 2\sin x$$

We have the following information about x and y

$$x(1) = 0, \quad x'(1) = 2, \quad y(1) = 3, \quad \text{and} \quad y'(1) = -1.$$

Find $\frac{dz}{dt}$ when $t = 1$.

this is a $\frac{dx}{dt}$ value

this is a $\frac{dy}{dt}$ value

Implicit diff:

$$\frac{d}{dt} z = \frac{d}{dt}\left(xy - 2\sin x\right)$$

product

$$\frac{dz}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt} - 2\cos x \cdot \frac{dx}{dt}$$

$$\frac{dz}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} - 2\frac{dx}{dt}\cos x$$

When $t=1$

$$\frac{dz}{dt} = 3(2) + 0(-1) - 2(2)\cos(0)$$

$$= 6-4 = 2$$

i.e. $z'(1) = 2$

Question

Suppose x and y depend on time t and $x^2 + y^2 = 5$. When $t = 2$

$$x = 1, \quad y = -2, \quad \text{and} \quad \frac{dy}{dt} = 3.$$

When $t = 2$, $\dfrac{dx}{dt} =$

$$\frac{d}{dt}\left(x^2+y^2\right) = \frac{d}{dt}5$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(1)\frac{dx}{dt} + 2(-2)(3) = 0$$

at some moment!

(a) $\dfrac{1}{2}$

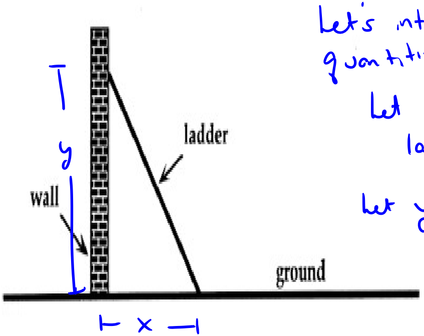
(b) 6

(c) $\dfrac{17}{2}$

(d) can't be determined without more information

Example

A 10 foot ladder rests against a wall. The base of the ladder begins to slide along the ground. At the moment when the base of the ladder is 6 feet from the wall, it is sliding at a rate of 1 inch per second. At what rate is the top of the ladder sliding down the wall at the moment when the base is 6 feet from the wall?



Let's introduce variables for changing quantities.

Let x be the distance between ladder base and wall in feet.

Let y be the distance between the top of the ladder and the ground also in feet.

We're given $\frac{dx}{dt} = 1 \, \frac{\text{in}}{\text{sec}}$ when $x = 6$ ft

$$= \frac{1}{12} \, \frac{\text{ft}}{\text{sec}}$$

The question is: What is $\frac{dy}{dt}$ when $x = 6$ ft?

From the geometry, $x^2 + y^2 = 10^2$ at all times.

Using implicit differentiation

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt} 10^2 = 0$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

we need to know y when $x = 6$ ft.

$$x^2 + y^2 = 10^2 \Rightarrow 6^2 + y^2 = 10^2 \Rightarrow y^2 = 100 - 36 = 64$$

$y = 8$ ft when $x = 6$ ft

At this moment

$$6\,\text{ft}\left(\frac{1}{12}\frac{\text{ft}}{\text{sec}}\right) + 8\,\text{ft}\,\frac{dy}{dt} = 0$$

$$\frac{1}{2}\frac{\text{ft}^2}{\text{sec}} + 8\,\text{ft}\,\frac{dy}{dt} = 0$$

$$8\,\text{ft}\,\frac{dy}{dt} = -\frac{1}{2}\,\frac{\text{ft}^2}{\text{sec}}$$

$$\frac{dy}{dt} = \frac{\frac{-1}{2}\,\frac{\text{ft}^2}{\text{sec}}}{8\,\text{ft}} = \frac{-1}{16}\,\frac{\text{ft}}{\text{sec}}$$

y is decreasing at a rate of $\frac{1}{16}$ ft per second.

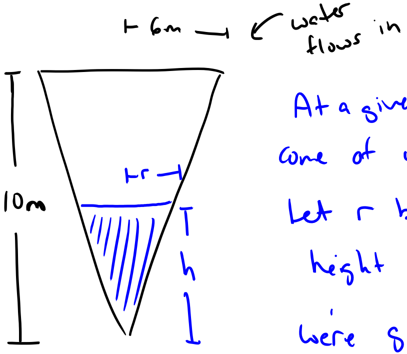
That is, the top is falling at a rate of $3/4$ inches per second at that moment.

General Approach to Solving Related Rates Prob.

- Identifty known and unknown quantities and assign variables.
- Create a diagram if possible.
- Use the diagram, physical science, and mathematics to connect known quantities to those being sought.
- **Relate the rates** of change using implicit differentiation.
- Substitute in known quantities and solve for desired quantities.

Example

A reservoir in the shape of an inverted right circular cone has height 10m and base radius 6m. If water is flowing into the reservoir at a constant rate of 50m^3/min. What is the rate at which the height of the water is increasing when the height is 5m?



At a given moment, there is a smaller cone of water inside the reservoir.

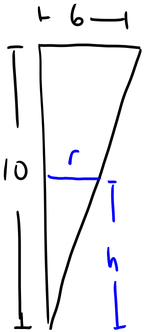
Let r be the base radius and h the height of the water, both in meters.

We're given a rate of change of Volume

$$\frac{dV}{dt} = 50 \frac{m^3}{min}$$ for volume V.

The question is: What is $\frac{dh}{dt}$ when $h = 5m$?

From geometry $V = \frac{\pi}{3} r^2 h$. We can reduce this to a function without r using similar triangles.



$$\frac{r}{h} = \frac{6}{10} = \frac{3}{5} \Rightarrow r = \frac{3}{5} h$$

So

$$V = \frac{\pi}{3}\left(\frac{3}{5}h\right)^2 h = \frac{\pi}{3}\,\frac{9h^2}{25}\,h$$

$$= \frac{3\pi}{25} h^3$$

Implicit diff:

$$\frac{d}{dt} V = \frac{d}{dt}\left(\frac{3\pi}{25} h^3\right)$$

$$\frac{dV}{dt} = \frac{3\pi}{25}(3h^2)\frac{dh}{dt} = \frac{9\pi}{25} h^2 \frac{dh}{dt}$$

When $h = 5\,m$

$$50 \frac{m^3}{min} = \frac{9\pi}{25}(5m)^2 \frac{dh}{dt}$$

$$= \frac{9\pi}{25}(25m^2)\frac{dh}{dt}$$

$$= 9\pi\, m^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{50\ \frac{m^3}{min}}{9\pi\ m^2} = \frac{50}{9\pi}\ \frac{m}{min}$$

The height is increasing at a rate of $\frac{50}{9\pi}$ m per minute when the height is 5 m.

Let's Do One Together

A child 1 m tall is walking near a street lamp that is 6 m tall. If she walks away from the street light at a constant rate of 20 m/min, how fast is her shadow lengthening?

Let's start with a diagram.

x and s are changing, their rates of change are $\frac{dx}{dt}$ and $\frac{ds}{dt}$

Take x and s in meters, time in minutes.

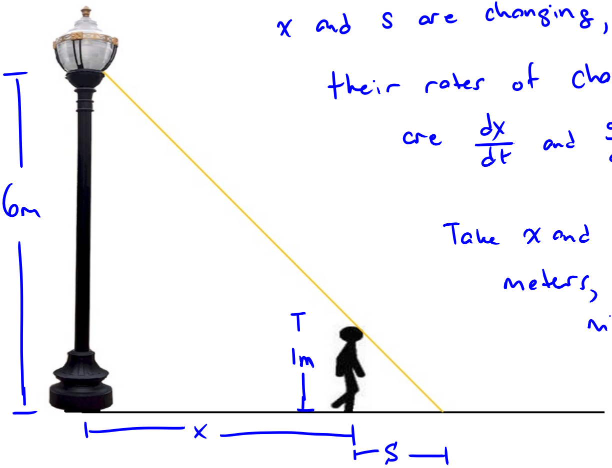
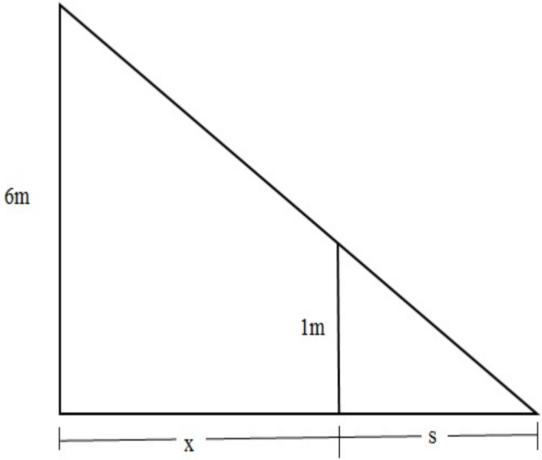


Figure: Let x be the child's distance from the lamp and s the length of her shadow.

Question

Making use of similar triangles, s and x are related by the equations



(a) $\frac{s}{1} = \frac{x}{6}$

(b) $\frac{s+x}{1} = \frac{x}{6}$

(c) $\frac{s}{1} = \frac{s+x}{6}$

(d) $\frac{s}{6} = \frac{s+x}{1}$

Question

Given the relationship $\frac{s}{1} = \frac{s+x}{6}$, $\frac{ds}{dt}$ is related to $\frac{dx}{dt}$ by

(a) $\frac{ds}{dt} = \frac{1}{5}\frac{dx}{dt}$

(b) $\frac{ds}{dt} = \frac{1}{6}\frac{dx}{dt}$

(c) $\frac{ds}{dt} = 5\frac{dx}{dt}$

(d) $\frac{ds}{dt} = 6\frac{dx}{dt}$

$6s = s + x$

$6s - s = x$

$5s = x$

$s = \frac{1}{5}x$

$\frac{ds}{dt} = \frac{1}{5}\frac{dx}{dt}$

Question

Recalling that the child was walking away at a constant rate of 20 m/min, how fast is her shadow lengthening?

(a) 20 m/min

(b) 3.33 m/min

(c) 5 m/min

(d) 4 m/min

$$\frac{dx}{dt} = 20 \ \frac{m}{min}$$

$$\frac{ds}{dt} = \frac{1}{5} \frac{dx}{dt} = \frac{1}{5}\left(20 \frac{m}{min}\right)$$

$$= 4 \ \frac{m}{min}$$