July 10 Math 2254 sec 001 Summer 2015

Section 8.5: Alternating Series and Absolute Convergence

Theorem: (The Alternating Series Test) Let $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ be an alternating series. If

$$(i) \quad \lim_{n\to\infty} a_n = 0$$

$$\text{and} \quad (ii) \quad a_{n+1} \le a_n \qquad \text{for all } n,$$

then the series is convergent.

Example

Determine the convergence or divergence of the series

(c) $\sum_{n=2}^{\infty} \cos(n\pi) \frac{n}{n^2 - 3}$

$= \sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2-3}$

n	$\cos(n\pi)$
2	$\cos(2\pi) = 1$
3	$\cos(3\pi) = -1$
4	$\cos(4\pi) = 1$
5	$\cos(5\pi) = -1$
	$\vdots$

$\cos(n\pi) = (-1)^n$

Here $a_n = \frac{n}{n^2-3}$

Alt. Series test

i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2-3} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1-\frac{3}{n^2}} = \frac{0}{1-0} = 0 \quad \checkmark$$

ii) Is $a_{n+1} \le a_n$? We'll use the related function $f(x) = \dfrac{x}{x^2-3}$

$$f'(x) = \frac{1(x^2-3) - x(2x)}{(x^2-3)^2} = \frac{x^2-3-2x^2}{(x^2-3)^2} = \frac{-(x^2+3)}{(x^2-3)^2} < 0$$

so f is decreasing.

$$a_{n+1} = f(n+1) < f(n) = a_n$$

Thus $a_{n+1} \leq a_n$ is true.

The series converges by the alternating series test.

An Observation

Note: If property (i) doesn't hold, i.e. if $\lim_{n\to\infty} a_n \neq 0$, then the series will definitely diverge by the divergence test.

However, if the first condition DOES hold, but the second does not, the test is inconclusive. The series may converge or it may diverge. Some other test must be used.

A Strange Case: ($a_{n+1} \leq a_n$ is required)

Consider the series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{where} \quad a_n = \begin{cases} \frac{4}{(n+1)^2}, & n \text{ odd} \\ \frac{2}{n}, & n \text{ even} \end{cases}$$

It is easy to see that $\lim_{n\to\infty} b_n = 0$. But note that the terms a_n are

$$\{a_n\} = \left\{1, 1, \frac{1}{4}, \frac{1}{2}, \frac{1}{9}, \frac{1}{3}, \frac{1}{16}, \frac{1}{4}, \dots\right\}$$

So that

$$\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is **divergent.**

Another Strange Case:

Consider the series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{where} \quad a_n = \begin{cases} \frac{4}{(n+1)^2}, & n \text{ odd} \\ \frac{8}{n^3}, & n \text{ even} \end{cases}$$

It is easy to see that $\lim_{n\to\infty} a_n = 0$. But note that the terms a_n are

$$\{a_n\} = \left\{1, 1, \frac{1}{4}, \frac{1}{8}, \frac{1}{9}, \frac{1}{27}, \frac{1}{16}, \frac{1}{64}, \frac{1}{25}, \frac{1}{125}, \frac{1}{36}, \ldots\right\}$$

So that

$$\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty}\frac{1}{n^3} - \sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is **convergent**.

Different Types of Convergence for Series with Mixed Signs

Note that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad \text{converges,}$$

but

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \quad \text{diverges.}$$

However, both

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots, \quad \text{and}$$

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \quad \text{converge.}$$

Absolute Convergence

Definition: A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

is convergent.

For Example: The series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} \quad \text{is } \textbf{absolutely convergent.}$$

The alternating harmonic series is **NOT** absolutely convergent.

Conditional Convergence

Definition: A series that is convergent but is not absolutely convergent is called **conditionally convergent**.

The alternating harmonic series IS conditionally convergent.

Theorem on Absolute Convergence

Theorem: If a series is absolutely convergent, it is convergent.

Remark: If we can show that a series is absolutely convergent, then we can conclude that it is convergent.

Remark: Of course, this doesn't mean that a series that isn't absolutely convergent must diverge. It may be conditionally convergent, and some effort may be required to determine its nature.

Example

Determine if the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$$

Let's consider $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^3} \right|$ which is a series of nonnegative terms.

So the direct comparison test can be used.

Since $|\sin n| \leq 1$, $\left| \frac{\sin n}{n^3} \right| \leq \frac{1}{n^3}$.

The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Hence

$\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^3} \right|$ converges by direct comparison.

And $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ is absolutely convergent.

Example

Determine if the series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

Alternating series test: $a_n = \dfrac{1}{n \ln n}$

i) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n \ln n} = 0$

ii) $\displaystyle a_{n+1} = \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n} = a_n$

Both properties hold; this series is convergent.

To classify convergence, consider

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Integral test: $f(x) = \frac{1}{x \ln x}$ is

positive, decreasing and continuous for $x \geq 2$.

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x} \frac{1}{\ln x}\,dx \; *$$

$$= \lim_{t\to\infty} \ln|\ln x| \Big|_2^t$$

$$= \lim_{t\to\infty} \left(\ln|\ln t| - \ln|\ln 2| \right) = \infty$$

The integral is divergent. So the series

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent.

As $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges and $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right|$

diverges, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

is conditionally convergent.

$$* \int \frac{1}{\ln x} \cdot \frac{1}{x} \, dx \qquad u = \ln x \qquad du = \frac{1}{x} dx$$

$$= \int \frac{1}{u} \, du = \ln|u| + C$$

$$= \ln|\ln x| + C$$

Section 8.6: The Ratio Test and The Root Test

In this section, we introduce two tests that may be used to conclude absolute convergence. We need not consider series with all one sign (all positive or all negative terms). But in general, we wish to consider any series

$$\sum_{n=1}^{\infty} a_n \quad \text{where } a_n \neq 0.$$

Theorem: The Ratio Test (a test for abs. convergence)

Theorem: Let $\sum a_n$ be a series of nonzero terms, and define the number L by

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

If

(i) $L < 1$, the series is absolutely convergent;

(ii) $L > 1$, the series is divergent;

(iii) $L = 1$, the test is inconclusive.

Remark: In the case $L = 1$, the series may be absolutely convergent, conditionally convergent, or divergent. This test truly **fails**, and some other test or analysis is necessary to draw any conclusion.

Examples

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{4^n}$

Use the ratio test: $a_n = (-1)^n \dfrac{n^2}{4^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(n+1)^2}{4^{n+1}}}{(-1)^n\frac{n^2}{4^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n+1}}(n+1)^2}{\cancel{4^{n}}4}\cdot\frac{\cancel{4^n}}{\cancel{(-1)^n}\,n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-1\,(n+1)^2}{4\,n^2}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{4n^2}$$

$$= \lim_{n\to\infty}\frac{1}{4}\left(\frac{n+1}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right)^2 = \frac{1}{4}(1+0)^2 = \frac{1}{4}$$

$$L = \frac{1}{4} < 1$$

The series converges absolutely by the ratio test.

Recall: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

This will come in handy for the next example.