July 11 Math 2306 sec 52 Summer 2016

Section 12: LRC Series Circuits

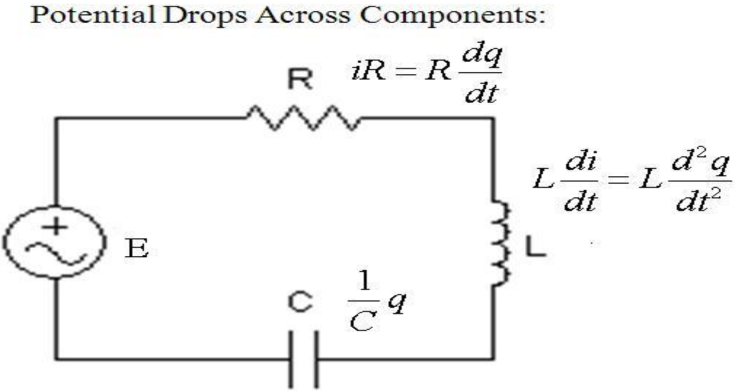


Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

With $R > 0$, $\lim_{t\to\infty} q_c(t) = 0$. Hence q_c is called the **transient state charge** of the system.

The function q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state charge of the system if the applied force is $E(t) = 5\cos(10t)$.

We found the equation in standard form to be

$$q'' + 20q' + 500q = 10\cos(10t)$$

whose complementary solution
$q_c = c_1 e^{-10t}\cos(20t) + c_2 e^{-10t}\sin(20t)$. We sought a particular solution of the form

$$q_p = A\cos(10t) + B\sin(10t)$$

and upon substitution into the ODE arrived at

$$[400A + 200B]\cos(10t) + [-200A + 400B]\sin(10t) = 10\cos(10t)$$

Matching

$$400A + 200B = 10$$
$$-200A + 400B = 0 \Rightarrow 200A = 400B \Rightarrow A = 2B$$

So $400(2B) + 200B = 10$

$$1000B = 10 \Rightarrow B = \frac{10}{1000} = \frac{1}{100}$$

$$A = 2\left(\frac{1}{100}\right) = \frac{1}{50}$$

The steady state Charge is

$$q_p = \frac{1}{50}\cos(10t) + \frac{1}{100}\sin(10t).$$

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s, t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_{\alpha}^{\beta} G(s, t)\, dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

For example, assume $s \neq 0$ and compute

$$\int_1^2 \cos(st)\,dt$$

Treat s as constant while integrating.

u-sub

let $u = st$

$du = s\,dt \Rightarrow dt = \frac{1}{s}du$

when $t=1$, $u=s$

$t=2$, $u=2s$

$$= \int_s^{2s} \cos(u)\frac{1}{s}\,du$$

$$= \frac{1}{s}\int_s^{2s} \cos u\,du$$

$$= \frac{1}{s}\sin u\Big|_s^{2s} = \frac{1}{s}\sin(2s) - \frac{1}{s}\sin(s)$$

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.

Note 2: If we take s to be real-valued, then

$$\lim_{t\to\infty} e^{-st} = 0 \quad \text{if } s > 0, \text{ and} \qquad \lim_{t\to\infty} e^{-st} = \infty \quad \text{if } s < 0.$$

Find the Laplace transform of $f(t) = 1$

By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \Rightarrow \mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt$

Consider the case $s=0$. The integral is

$$\int_0^\infty dt = \lim_{b\to\infty} \int_0^b dt = \lim_{t\to\infty} t\Big|_0^b = \lim_{b\to\infty} (b-0) = \infty \quad \text{divergent}$$

$s=0$ is not in the domain of $\mathcal{L}\{1\}$.

For $s \neq 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} \frac{1}{-s} e^{-st} \Big|_0^b$$

$$= \lim_{b\to\infty} \frac{1}{-s}\left(e^{-sb} - e^{0}\right) = \frac{1}{-s}(0-1) \quad \text{if } s>0$$

Diverges if $s<0$

$$= \frac{1}{s}$$

So $\mathcal{L}\{1\} = \frac{1}{s}$ with domain $s>0$.

We'll forgo the formal limit taking notation from now on.

Find the Laplace transform of $f(t) = t$

By definition $\mathscr{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

If $s=0$, the integral is $\int_0^\infty t\, dt = \frac{t^2}{2}\Big|_0^\infty = \infty$ divergent

$s=0$ is not in the domain.

For $s \neq 0$

$\mathscr{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

By parts

$u = t \qquad du = dt$

$v = \frac{-1}{s} e^{-st} \qquad dv = e^{-st}\, dt$

$= \frac{-1}{s} t e^{-st}\Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt$

If $s<0$, the integral diverges

For $s>0$

$$\mathcal{L}\{t\} = \frac{-1}{s} t e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{-1}{s}(0-0) + \underbrace{\frac{1}{s}\int_0^\infty e^{-st}\,dt}_{\mathcal{L}\{1\}}$$

$$= \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

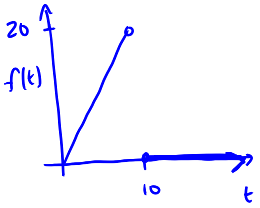
Hence $\mathcal{L}\{t\} = \frac{1}{s^2}$ with domain $s>0$.

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

Plot of f



By definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st}(2t)\,dt + \int_{10}^\infty e^{-st}\cdot 0\,dt \quad = \int_0^{10} 2e^{-st}t\,dt$$

If $s=0$, $\int_0^{10} 2t\,dt = t^2\Big|_0^{10} = 100-0 = 100$

For $s\neq 0$ $\int_0^{10} 2e^{-st}t\,dt =$

Use parts from the last example

$= 2\left[\frac{-1}{s}te^{-st}\Big|_0^{10} + \frac{1}{s}\int_0^{10} e^{-st}\,dt\right]$

$= 2\left[\frac{-1}{s}te^{-st}\Big|_0^{10} + \frac{1}{s}\left(\frac{-1}{s}e^{-st}\right)\Big|_0^{10}\right.$

$= 2\left[\frac{-1}{s}(10)e^{-10s} - \frac{-1}{s}\cdot 0\,e^{0} + \frac{1}{s}\left(\frac{-1}{s}e^{-10s} - \frac{-1}{s}e^{0}\right)\right]$

$$= 2\left[\frac{-10}{s}e^{-10s} - \frac{1}{s^2}e^{-10s} + \frac{1}{s^2}\right]$$

$$= \frac{2}{s^2} - \frac{20}{s}e^{-10s} - \frac{2}{s^2}e^{-10s}$$

$$\mathcal{L}\{f(t)\} = \begin{cases} 100, & s = 0 \\ \frac{2}{s^2} - \frac{20}{s}e^{-10s} - \frac{2}{s^2}e^{-10s}, & s \neq 0 \end{cases}$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0 \text{ for } n = 1, 2, \dots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Examples: Evaluate (the Laplace transform of)

(a) $f(t) = \cos(\pi t)$

Use $\mathcal{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$

So $\mathcal{L}\{\cos(\pi t)\} = \dfrac{s}{s^2+\pi^2}$

Examples: Evaluate

Use $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ $s>0$

$\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ $s>a$

$\mathcal{L}\{1\} = \dfrac{1}{s}$ $s>0$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{2t^4 - e^{-5t} + 3\} = \mathcal{L}\{2t^4\} + \mathcal{L}\{-e^{-5t}\} + \mathcal{L}\{3\}$$

$$= 2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$= 2\,\frac{4!}{s^{4+1}} - \frac{1}{s-(-5)} + 3\,\frac{1}{s}$$

$$= \underset{s>0}{\frac{48}{s^5}} - \underset{s>-5}{\frac{1}{s+5}} + \underset{s>0}{\frac{3}{s}}, \quad s>0$$

for all three to hold

Examples: Evaluate

(c) $f(t) = (2-t)^2$ $= 4-4t+t^2$

Expand the square to use the table

$$\mathscr{L}\{1\} = \frac{1}{s}$$

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathscr{L}\{(2-t)^2\} = \mathscr{L}\{4-4t+t^2\}$$

$$= 4\mathscr{L}\{1\} - 4\mathscr{L}\{t\} + \mathscr{L}\{t^2\}$$

$$= 4\cdot\frac{1}{s} - 4\,\frac{1!}{s^{1+1}} + \frac{2!}{s^{2+1}}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

Examples: Evaluate

Recall $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

take $\theta = 5t$

(d) $f(t) = \sin^2 5t$

$$f(t) = \frac{1}{2} - \frac{1}{2}\cos(10t)$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sin^2(5t)\} = \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(10t)\right\}$$

$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos(10t)\}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+10^2} = \frac{1}{2s} - \frac{1}{2}\frac{s}{(s^2+100)}$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order* c provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0, \infty)$ and of exponential order c for some $c > 0$, then f has a Laplace transform for $s > c$.

An example of a function that doesn't have a Laplace transform is $f(t) = e^{t^2}$. This function grows faster than every exponential function* e^{ct}.

*The graph of $y = x^2$ is above every line $y = cx$ for all $x > 1$.

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \ldots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

$$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^7}\right\}$

Note if $n=6$, then $n+1=7$

$$\frac{1}{s^7} = \frac{6!}{6!}\cdot\frac{1}{s^7} = \frac{1}{6!}\,\frac{6!}{s^7}$$

So $\mathscr{L}^{-1}\left\{\dfrac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{6!}\,\dfrac{6!}{s^{6+1}}\right\} = \dfrac{1}{6!}\,\mathscr{L}^{-1}\left\{\dfrac{6!}{s^{6+1}}\right\} = \dfrac{1}{6!}\,t^6$

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$$

$$\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{3}\,\frac{3}{s^2+3^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

We need a partial fraction decomp on

$$\frac{s-8}{s^2-2s}$$

$$\frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$ Clear fractions

$$s-8 = A(s-2) + Bs$$

Set $s=0$ $\quad -8 = A(-2) + B(0) \Rightarrow -8 = -2A, \quad A = 4$

$s=2$ $\quad 2-8 = A(0) + B(2) \Rightarrow -6 = 2B, \quad B = -3$

So $\mathscr{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathscr{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$

$$= 4\,\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 3\,\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4(1) - 3e^{2t}$$

$$= 4 - 3e^{2t}$$