July 12 Math 1190 sec. 51 Summer 2017

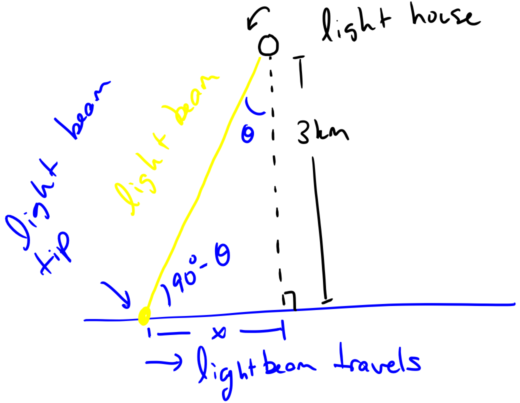
Section 4.1: Related Rates

General Approach to Solving Related Rates Problems:

- Identifty known and unknown quantities and assign variables.
- Create a diagram if possible.
- Use the diagram, physical science, and mathematics to connect known quantities to those being sought.
- **Relate the rates** of change using implicit differentiation.
- Substitute in known quantities and solve for desired quantities.

Example

A lighthouse is 3 km from a straight shoreline. Its light makes one revolution every 8 seconds. How fast is the light moving along the shoreline when it makes an angle of 30° with the shoreline?



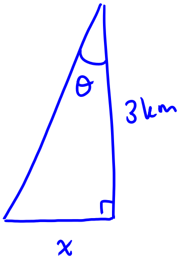
Let x be the distance of the light beam tip from the straight line between the light house and the shore.

Let θ be the angle between the light beam and the line between light house and shore.

The question is: What is $\frac{dx}{dt}$ when $90° - \theta = 30°$?

One revolution every 8 seconds gives a rate of change of θ.

$$\frac{d\theta}{dt} = \frac{1 \text{ rev}}{8 \text{ sec}} = \frac{2\pi \text{ rad}}{8 \text{ sec}} = \frac{\pi}{4} \frac{\text{rad}}{\text{sec}}$$



Of the six trig functions, the tangent is convenient

$$\tan\theta = \frac{x}{3\text{km}}$$

$$\Rightarrow x = 3\tan\theta$$

Differentiate: $\frac{d}{dt} x = \frac{d}{dt} 3\tan\theta$

$$\frac{dx}{dt} = 3\sec^2\theta \cdot \frac{d\theta}{dt}$$

When the angle with the shore is 30°, $\theta = 60^\circ$.

Note $\sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{1/2} = 2$

When $\theta = 60^\circ$

a.k.a $\frac{\text{rad}}{\text{sec}}$

$$\frac{dx}{dt} = 3\text{ km}\,(2)^2 \cdot \frac{\pi}{4}\,\frac{1}{\text{sec}}$$

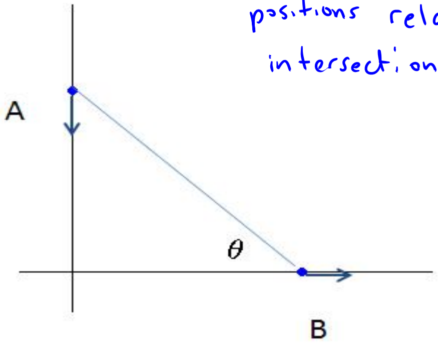
$$\frac{dx}{dt} = 12\left(\frac{\pi}{4}\right)\frac{\text{km}}{\text{sec}} = 3\pi\ \frac{\text{km}}{\text{sec}}$$

The beam is traveling along the shore at a rate of 3π km/sec at that moment.

Let's Do One Together

Pedestrians A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2m/sec, and B moves away from the intersection at 1m/sec. Our goal is to determine the rate at which the angle θ shown in the diagram is changing when A is 10m from the intersection and B is 20 m from the intersection?

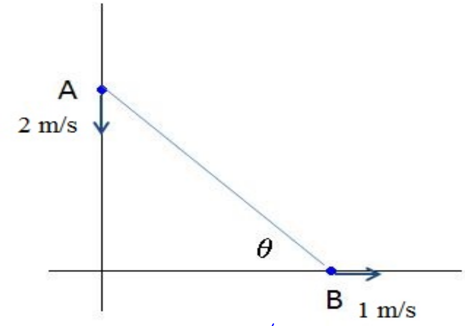
We'll call the pedestrians' positions relative to the intersection A and B.



Question

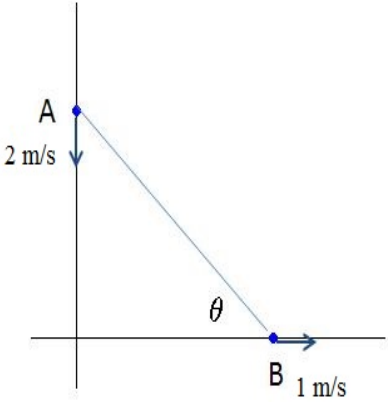
Let $A(t)$ be pedestrian A's position (distance to intersection), and $B(t)$ be pedestrian B's position. Let's make some observations:

(a) **True or False** A is decreasing.
(b) **True or False** B is increasing.



Question

From the diagram, which of the following are the rates of change of A and B (in m/s)?



(a) $\frac{dA}{dt} = -2$ and $\frac{dB}{dt} = 1$

(b) $\frac{dA}{dt} = 2$ and $\frac{dB}{dt} = -1$

(c) $\frac{dA}{dt} = -2$ and $\frac{dB}{dt} = -1$

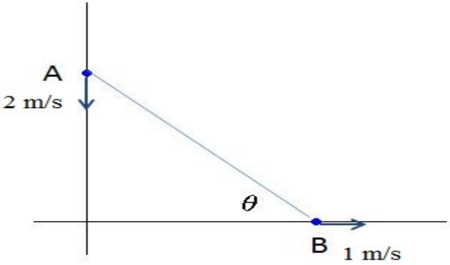
(d) $\frac{dA}{dt} = 2$ and $\frac{dB}{dt} = 1$

Relating the Rates

The pedestrians' positions and the intersection form a right triangle. So θ, A, and B are related by the equation

$$\tan\theta = \frac{A}{B}$$

Question: Use implicit differentiation to find an expression relating $\frac{d\theta}{dt}$ to the rates of A and B.



Question $\tan\theta = \frac{A}{B}$

The relation between the rates is given by

$\tan\theta = \dfrac{A}{B}$

(a) $$\frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$$

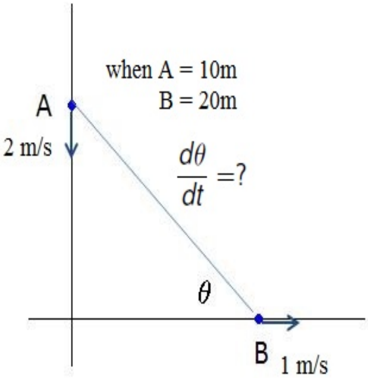
(b) $$\sec^2\left(\frac{d\theta}{dt}\right) = \frac{\frac{dA}{dt}}{\frac{dB}{dt}}$$

(c) $$\sec^2(\theta)\frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$$

(d) $$\sec^2(\theta)\frac{d\theta}{dt} = \frac{A}{B}\frac{dA}{dt} + \frac{A}{B}\frac{dB}{dt}$$

The Final Result

Determine the rate at which the angle θ shown in the diagram is changing when A is 10m from the intersection and B is 20 m from the intersection?

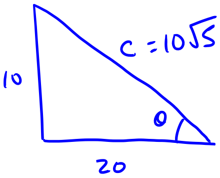


$$\sec^2\theta \, \frac{d\theta}{dt} = \frac{\frac{dA}{dt} B - A \frac{dB}{dt}}{B^2}$$

$$\frac{d\theta}{dt} = \frac{\frac{dA}{dt} B - A \frac{dB}{dt}}{B^2 \sec^2\theta}$$

$$\frac{dA}{dt} = -2 \ \frac{m}{s} \qquad \frac{dB}{dt} = 1 \ \frac{m}{sec}$$

To get the $\sec^2\theta$



$$c^2 = 10^2 + 20^2 = 100 + 400$$

$$= 500 \Rightarrow c = \sqrt{500} = 10\sqrt{5}$$

so $\sec\theta = \dfrac{10\sqrt{5}}{20} = \dfrac{\sqrt{5}}{2}$

When $A = 10$ and $B = 20$

$$\frac{d\theta}{dt} = \frac{-2\,\frac{m}{sec}\,(20\,m) - (10\,m)\left(1\,\frac{m}{sec}\right)}{(20\,m)^2\left(\frac{\sqrt{5}}{2}\right)^2}$$

$$\frac{d\theta}{dt} = \frac{-50\ \frac{m^2}{sec}}{400\,m^2\left(\frac{5}{4}\right)} = \frac{-10\ \frac{m^2}{sec}}{100\ m^2}$$

$$= \frac{-1}{10}\ \frac{1}{sec}$$

The angle is decreasing at a rate of $\frac{1}{10}$ radian per second.

Section 4.7: Optimization

Optimization problems arise in every field of study and every industry.

- minimize cost and maximize revenue,
- maximize crop yield,
- minimize driving time,
- maximize volume,
- minimize energy

Often, some constraint (extra condition) must simultaneously be satisfied.

Applied Optimization Example

← area is fixed, so this is a constraint

A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of its sides. What dimensions of the outer rectangle will require the smallest total length of fencing and how much fencing will be needed?

minimize fence length

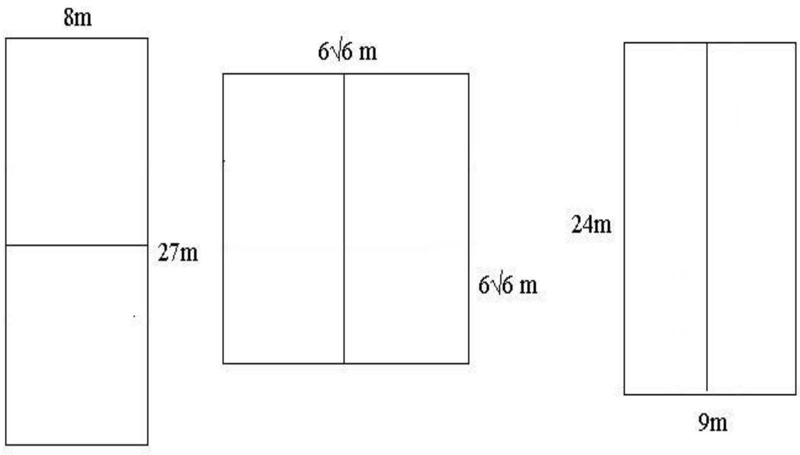
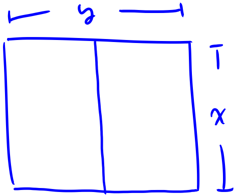


Figure: Different pea patch configuration that all enclose $216m^2$.

Consider a representative pea patch



It has 2 dimensions, length and width. Let x be the width and y the length in meters.

The area $A = xy$

The amount of fencing $F = 2y + 3x$

The question is: What values of x and y minimize F given that $A = 216\ m^2$?

To minimize F, we need it as a function of one variable. From $xy = 216$, $x = \frac{216}{y}$

Thus $F = 2y + 3\left(\frac{216}{y}\right) = 2y + \frac{3(216)}{y}$

Find crit. #s $\frac{dF}{dy} = 2 + 3(216)(-y^{-2}) = 2 - \frac{3(216)}{y^2}$

$\frac{dF}{dy}$ is defined for all $y>0$.

$$\frac{dF}{dy}=0 \implies 0=2-\frac{3(216)}{y^2} \implies 2=\frac{3(216)}{y^2}$$

$$\implies y^2=\frac{3(216)}{2}=324$$

$y=18$ or $y=-18$, but $y>0$ so we only get one critical number, $y=18$.

Let's verify that $y=18$ minimizes F.

2nd der. test : $\frac{d^2F}{dy^2} = 3(216)\left(-y^{-3}(-2)\right)$

$$\frac{d^2F}{dy^2} = \frac{3(2)(216)}{y^3}$$

$$F''(18) = \frac{3(2)(216)}{18^3} > 0$$

18 minimizes F.

When $y = 18$ m, $x = \frac{216 \text{ m}^2}{18 \text{ m}} = 12$ m

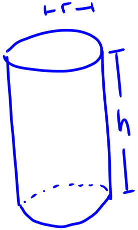
The pen should be 12m × 18 m with the extra piece of fencing 12 m.

Let's Do One Together

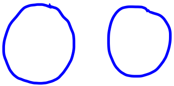
A can in the shape of a right circular cylinder is to have a volume of 128π cubic cm. The material that the top and bottom are made of costs \$0.20/cm^2 and the material that the lateral surface is made of costs \$0.10/cm^2. Find the dimensions of the can that minimize the total cost of production.

We need a cost function.

Let r and h be the base radius and height in cm.



There are 3 pieces

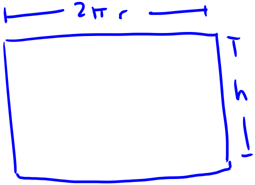


2 identical disks of area

$A_c = \pi r^2$

And a lateral piece

$$A_\ell = 2\pi r h$$



Question

The total cost $C =$ (cost of lateral surface) + (cost of top & bottom). The cost for the lateral surface was \$0.10/cm^2 while the cost for the top and bottom material is \$0.20/cm^2. The surface area was $S = 2\pi rh + 2\pi r^2$. Which of the following is the cost function?

(a) $C = 2\pi rh + 2\pi r^2$

(b) $C = 0.1(2\pi rh) + 0.2(2\pi r^2)$

(c) $C = 0.2(2\pi rh) + 0.1(2\pi r^2)$

(d) $C = (0.1)(0.2)(2\pi rh + 2\pi r^2)$

Question

The cost appears as a function of two variables, r and h. But we need it to be a function of only one variable.

The volume of the can $V = \pi r^2 h$. We are told it must hold 128π cm^3. Which of the following could be used to express C as a function of r alone?

$\pi r^2 h = 128\pi$

$r^2 h = 128$

$h = \frac{128}{r^2}$ or $r^2 = \frac{128}{h} \Rightarrow r = \sqrt{\frac{128}{h}}$

(a) $h = \dfrac{128}{r}$

(b) $r = \dfrac{128}{\sqrt{h}}$

(c) $h = \dfrac{128}{r^2}$

Question

We can write the cost function in terms of r as

$$C = \frac{25.6\pi}{r} + 0.4\pi r^2$$

Which of the following is the derivative of C with respect to r?

(a) $\dfrac{dC}{dr} = \dfrac{-25.6\pi}{r^2} + 0.8\pi r$

(b) $\dfrac{dC}{dr} = \dfrac{-25.6\pi + 0.8\pi r}{r^2}$

(c) $\dfrac{dC}{dr} = \dfrac{-25.6\pi}{r^2} + 0.4\pi r$

Question

$$\text{Given that} \quad \frac{dC}{dr} = \frac{-25.6\pi}{r^2} + 0.8\pi r,$$

The critical number(s) of C are

(a) 0 and 32

(b) 0 and $\sqrt[3]{32}$

zero is not in the domain of C

(c) can't be determined without more information

(d) $\sqrt[3]{32}$

$$\frac{25.6\pi}{r^2} = 0.8\pi r$$

$$\frac{25.6\pi}{0.8\pi} = r^3$$

$$r^3 = 32$$

Question

We suspect that the optimal size for the radius, the one that minimizes cost is $\sqrt[3]{32}$. We decide to use the second derivative test to check. We find that

$$\frac{d^2C}{dr^2} = \frac{d}{dr}\left(\frac{-25.6\pi}{r^2} + 0.8\pi r\right) = \frac{51.2\pi}{r^3} + 0.8\pi$$

With no computation, we determine that $r = \sqrt[3]{32}$ is a local minimum because

(a) $C''(r)$ is positive for all positive r, so the graph is concave up.

(b) $C''(r)$ is negative for all positive r, so the graph is concave up.

(c) $C''(r)$ is positive for all positive r, so the graph is concave down.

(d) $C''(r)$ is negative for all positive r, so the graph is concave down.

Question

Since the optimal $r = \sqrt[3]{32}$ and $h = 128/r^2$ our recommendation for minimizing the cost is a can with dimensions

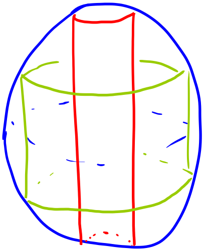
(a) radius of $\sqrt[3]{32}$ cm and height $128/\sqrt[3]{32}$ cm

(b) radius of $\sqrt[3]{32}$ cm and height $128/\sqrt[3]{32^2}$ cm

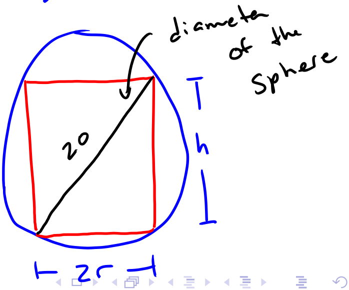
(c) radius of $\sqrt[3]{32}$ cm and height 4 cm

Applied Optimization Example

Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius 10.



If we cut the sphere in half through the poles on the axis of symmetry of the cylinder, we see a rectangle inside of a circle



Let r and h be the radius and height of the cylinder.

The volume $V = \pi r^2 h$.

By the pythagorean theorem

$$(2r)^2 + h^2 = 20^2$$

Task: maximize $V = \pi r^2 h$ subject to $4r^2 + h^2 = 20^2$

$$4r^2 + h^2 = 20^2 \implies r^2 = \frac{1}{4}(20^2 - h^2)$$

$$V = \pi r^2 h = \pi\left(\frac{1}{4}(20^2 - h^2)\right)h = \frac{\pi}{4}(20^2 h - h^3)$$

Find critical number(s)

$$V'(h) = \frac{\pi}{4}\left(20^2 - 3h^2\right)$$

$V'(h)$ is always defined.

$$V'(h) = 0 \Rightarrow \frac{\pi}{4}\left(20^2 - 3h^2\right) = 0$$

$$20^2 = 3h^2 \Rightarrow h^2 = \frac{20^2}{3}$$

$$h = \frac{20}{\sqrt{3}} \quad \text{or} \quad h = \frac{-20}{\sqrt{3}}$$ ← ignore since $h \geq 0$

We can test to see if $h = \frac{20}{\sqrt{3}}$ maximizes V

$$V''(h) = -\frac{6\pi}{4}h$$

$$V''\left(\frac{20}{\sqrt{3}}\right) = -\frac{6\pi}{4}\left(\frac{20}{\sqrt{3}}\right) < 0 \quad \text{concave down}$$

So V is maximum when $h = \frac{20}{\sqrt{3}}$.

The max volume is

$$V\left(\frac{20}{\sqrt{3}}\right) = \frac{\pi}{4}\left(20^2\left(\frac{20}{\sqrt{3}}\right) - \left(\frac{20}{\sqrt{3}}\right)^3\right)$$

$$= \frac{\pi}{4}\left(\frac{20^3}{\sqrt{3}} - \frac{20^3}{3\sqrt{3}}\right)$$

$$= \frac{\pi}{4} \frac{20^3}{\sqrt{3}} \left(1 - \frac{1}{3} \right)$$

$$= \frac{\pi}{4} \frac{20^3}{\sqrt{3}} \left(\frac{2}{3} \right) = \frac{\pi}{2} \frac{20^3}{3\sqrt{3}}$$

$$V\left(\frac{20}{\sqrt{3}}\right) = \frac{4000\pi}{3\sqrt{3}}$$