July 13 Math 2254 sec 001 Summer 2015

Section 8.6: The Ratio Test and The Root Test

Theorem: (The Ratio Test) Let $\sum a_n$ be a series of nonzero terms, and define the number L by

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

If

(i) $L < 1$, the series is absolutely convergent;

(ii) $L > 1$, the series is divergent;

(iii) $L = 1$, the test is inconclusive.

Useful Result

For the next example, the following limit is useful to know.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

This is covered in section 3.3 in Sullivan and Miranda. A generalization of this is

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

which may be familiar from problems involving compounded interest.

Examples

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Ratio test: $a_n = \frac{n^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n \cdot \cancel{(n+1)}}{\cancel{n!}\,\cancel{(n+1)}} \cdot \frac{\cancel{n!}}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$L = e > 1$ The series is divergent by the ratio test.

Side note: recall $n! = 1\cdot 2\cdots n$

Note $(2n)! = 1\cdot 2\cdot 3\cdots n\cdot(n+1)\cdot(n+2)\cdots(2n)$

$(2(n+1))! = (2n+2)!$

$= 1\cdot 2\cdot 3\cdots n\cdots(2n)\cdot(2n+1)(2n+2)$

Examples

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n)!}$

Ratio test: $a_n = \dfrac{(-1)^n \pi^{2n}}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n}+1}\pi^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{(\cancel{-1})^{n}\pi^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)\pi^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{\pi^{2n}}\right|$$

$$= \lim_{n\to\infty} \frac{\cancel{\pi^{2n}} \cdot \pi^2 \quad \cancel{(2n)!}}{\cancel{(2n)!}(2n+1)(2n+2)\,\cancel{\pi^{2n}}}$$

$$= \lim_{n\to\infty} \frac{\pi^2}{(2n+1)(2n+2)} = 0$$

$L = 0 < 1.$ The series is absolutely convergent by the ratio test.

Examples

Use the ratio test to determine the values of t for which the series is guaranteed to be absolutely convergent.

$$\sum_{n=0}^{\infty} 4^n t^n = 4^0 t^0 + 4^1 t^1 + 4^2 t^2 + \dots$$

$$= 1 + 4t + 16t^2 + \dots$$

If $t=0$, the series is obviously convergent w/ sum 1.

For $t \neq 0$, use the ratio test w/ $a_n = 4^n t^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{4^{n+1}}\,\cancel{t^{n+1}}}{\cancel{4^n}\,\cancel{t^n}}\right|$$

$$= \lim_{n\to\infty}|4t| = |4t|$$

$$L = |4t| = 4|t|$$

The series is absolutely convergent if

$$4|t| < 1$$

$$\implies \quad |t| < \frac{1}{4}$$

$$\implies \quad -\frac{1}{4} < t < \frac{1}{4}$$

The series is guaranteed to be absolutely convergent if t is in the interval $\left(-\frac{1}{4}, \frac{1}{4}\right)$

Ratio Test Failure

Apply the ratio test to the known **divergent** series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Ratio test: $a_n = \frac{1}{n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$$

Ratio test fails

Ratio Test Failure

Apply the ratio test to the known **convergent** series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Ratio test: $a_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right|$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1^2 = 1$$

Ratio test fails.

Theorem: The Root Test

Theorem: Let $\sum a_n$ be a series of nonzero terms, and define the number L by

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

If

(i) $L < 1$, the series is absolutely convergent;

(ii) $L > 1$, the series is divergent;

(iii) $L = 1$, the test is inconclusive.

Examples

Determine if the series is absolutely convergent, conditionally convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^{2n}}{n^n}$

Root test : $a_n = \dfrac{\pi^{2n}}{n^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{\pi^{2n}}{n^n}\right|}$$

$$= \lim_{n\to\infty} \left(\frac{\pi^{2n}}{n^n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\pi^{2n\cdot\frac{1}{n}}}{n^{n\cdot\frac{1}{n}}} = \lim_{n\to\infty} \frac{\pi^2}{n} = 0$$

$L = 0 < 1$, the series is absolutely convergent by the root test.

(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+1}{2n-1}\right)^n$

Root test: $a_n = \left(\frac{3n+1}{2n-1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3n+1}{2n-1}\right)^n\right|}$$

$$= \lim_{n\to\infty} \left(\left(\frac{3n+1}{2n-1}\right)^n\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{3n+1}{2n-1}$$

$$= \lim_{n\to\infty} \frac{3n+1}{2n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{3+\frac{1}{n}}{2-\frac{1}{n}} = \frac{3+0}{2-0} = \frac{3}{2}$$

$$L = \frac{3}{2} > 1$$

The series diverges by the root test.

Section 8.7: Summary of Tests for Series

Potentially Useful Guidelines for Analyzing a Series $\sum a_n$

★ Does it have a specific *type*? (p-series, geometric, telescoping, alternating)

★ If you can readily see that $\lim_{n\to\infty} a_n \neq 0$, use the Divergence test.

★ If $a_n > 0$ and the function $f(n) = a_n$ looks like you can integrate it (i.e. $\int_1^\infty f(x)\,dx$ is manageable), try the integral test.

★ If it involves a rational function in n or a ratio of roots and powers of n, a direct or limit comparison test (comparing to a p-series) might be useful.

★ If it looks very similar to a geometric series, but is not quite a geometric series, a direct or limit comparison test to a geometric may be useful.

★ If it involves factorials or complicated products, the ratio test might lead to the necessary conclusion. If it involves expressions to the n^{th} power, the root test may work.

◇ Remember that the ratio & root tests (when conclusive) determine absolute convergence. When using the alternating series test, if a series is found to be convergent remember to check for absolute convergence.

Examples

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{m=0}^{\infty} \frac{2^m}{3^m + 5^m}$

Root test: $a_m = \dfrac{2^m}{3^m + 5^m}$

$$\lim_{m\to\infty} \sqrt[m]{|a_m|} = \lim_{m\to\infty} \sqrt[m]{\left|\frac{2^m}{3^m+5^m}\right|}$$

$$= \lim_{m\to\infty} \left(\frac{2^m}{3^m+5^m}\right)^{\frac{1}{m}} = \lim_{m\to\infty} \frac{2}{(3^m+5^m)^{\frac{1}{m}}}$$

Let's try something else.

Note for each n

$$\frac{2^n}{3^n + 5^n} < \frac{2^n}{5^n}$$

Since $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$ is a convergent

geometric series.

The series converges by direct

comparison.

Note $\displaystyle\sum_{m=0}^{\infty} \left| \frac{2^m}{3^m + 5^m} \right| = \sum_{m=0}^{\infty} \frac{2^m}{3^m + 5^m}$

all positive terms

So the series is absolutely convergent.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-3)^k}{k!} = \sum_{k=1}^{\infty} \frac{(-1)^k 3^k}{k!}$

Alt. Series test

$a_k = \dfrac{3^k}{k!}$

i) $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{3^k}{k!} \leq \lim_{k\to\infty} \frac{27}{2k} = 0$

$$\frac{3^k}{k!} = \frac{3\cdot 3\cdot 3\cdots 3}{1\cdot 2\cdot 3\cdots k} = 3\left(\frac{3}{2}\right)(1)\cdot\frac{3}{4}\cdot\frac{3}{5}\cdots\frac{3}{k}$$

$$\leq \frac{9}{2}\cdot 1\cdot 1\cdots\frac{3}{k} = \frac{27}{2k}$$

ii) Is $a_{k+1} \leq a_k$?

$$a_{k+1} = \frac{3^{k+1}}{(k+1)!} = \frac{3 \cdot 3^k}{(k+1)k!} = \frac{3}{k+1} \cdot \left(\frac{3^k}{k!}\right)$$

If $k \geq 3$ then $\frac{3}{k+1} \leq \frac{3}{4} < 1$

So for $k \geq 3$ $\quad a_{k+1} = \frac{3}{k+1}\left(\frac{3^k}{k!}\right) < \frac{3^k}{k!} = a_k$

Both (i) and (ii) hold. The series is convergent by the alternating series test.

Consider $\displaystyle\sum_{k=1}^{\infty} \left| \frac{(-1)^k 3^k}{k!} \right| = \sum_{k=1}^{\infty} \frac{3^k}{k!}$

Ratio test: $a_k = \dfrac{3^k}{k!}$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{3^{k+1}}{(k+1)!} \cdot \frac{k!}{3^k} \right|$$

$$= \lim_{k\to\infty} \frac{\cancel{3^k} \cdot 3 \quad \cancel{k!}}{\cancel{k!}\,(k+1) \quad \cancel{3^k}}$$

$$= \lim_{k\to\infty} \frac{3}{k+1} = 0 \qquad L = 0 < 1$$

This series converges too.

Hence $\sum_{k=1}^{\infty} \frac{(-1)^k 3^k}{k!}$ converges

absolutely.