July 13 Math 2306 sec 52 Summer 2016

Section 13 & 14 : The Laplace Transform & Inverse Laplace Transforms

Recall that if f is piecewise continuous on $[0, \infty)$ and of exponential order c for some $c > 0$, then the Laplace transform of f is defined by

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Moreover, if $F(s) = \mathscr{L}\{f(t)\}$, then we say that $f(t)$ is an inverse Laplce transform of $F(s)$ and write

$$f(t) = \mathscr{L}^{-1}\{F(s)\}.$$

Basic Table of Laplace Transforms

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Evaluate the Transform or Inverse Transform

(a) $\mathscr{L}\{2t^3 - \sin(5t)\} = 2\mathscr{L}\{t^3\} - \mathscr{L}\{\sin(5t)\}$

$$= 2\,\frac{3!}{s^4} - \frac{5}{s^2+5^2}$$

$$= \frac{12}{s^4} - \frac{5}{s^2+25}$$

Examples: Evaluate

(b) $\mathscr{L}^{-1}\left\{\frac{4s}{s^2+8}\right\} = 4\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+8}\right\}$

$= 4\cos(\sqrt{8}\,t)$

Examples: Evaluate

(c) $\mathscr{L}\left\{(e^{2t}-1)^2\right\} = \mathscr{L}\left\{e^{4t} - 2e^{2t} + 1\right\}$

$$= \mathscr{L}\{e^{4t}\} - 2\mathscr{L}\{e^{2t}\} + \mathscr{L}\{1\}$$

$$= \frac{1}{s-4} - 2\frac{1}{s-2} + \frac{1}{s}$$

$$= \frac{1}{s-4} - \frac{2}{s-2} + \frac{1}{s}$$

Examples: Evaluate

(d) $\mathscr{L}^{-1}\left\{\dfrac{2}{s^2+3s}\right\}$

Partial fractions

$$\frac{2}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$ Mult. by $s(s+3)$

$$2 = A(s+3) + Bs$$

Set $s=0$ $\quad 2 = A(3) + B\cdot 0 \Rightarrow A = \frac{2}{3}$

$s=-3$ $\quad 2 = A(0) + B(-3) \Rightarrow B = \frac{-2}{3}$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+3s}\right\} = \mathcal{L}^{-1}\left\{\frac{2/3}{s} - \frac{2/3}{s+3}\right\}$$

$$= \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{2}{3} - \frac{2}{3}e^{-3t}$$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

Note that by definition

$$\mathscr{L}\left\{e^t t^2\right\}=\int_0^\infty e^{-st}e^t t^2\,dt = \int_0^\infty e^{-(s-1)t}\,t^2\,dt$$

$$e^{-st}\cdot e^{t} = e^{-st+t} = e^{(-s+1)t} = e^{-(s-1)t}$$

Let $w = s-1$

$$= \int_0^\infty e^{-wt}\,t^2\,dt$$

$\mathscr{L}\{t^2\}$ evaluated at w

$$= \frac{2}{w^3}$$

So $\mathcal{L}\{e^t t^2\} = \frac{2}{w^3}$ where $w = s-1$

i.e. $\mathcal{L}\{e^t t^2\} = \frac{2}{(s-1)^3}$

If $F(s) = \mathcal{L}\{t^2\}$ then

$$F(s-1) = \mathcal{L}\{e^t t^2\}$$

Theorem (translation in s)

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

We'll use this primarily with irreducible quadratic factors or repeated linear factors in the denominator.

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 doesn't factor

Its discriminant is

$2^2-4\cdot1\cdot2 = 4-8 < 0$

we'll complete the square

$$s^2+2s+2 = s^2+2s+1+2-1 = (s+1)^2+1$$

So $$\frac{s}{s^2+2s+1} = \frac{s}{(s+1)^2+1} = \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

form $\frac{s}{s^2+1}$ $\qquad \frac{1}{s^2+1}$

$$= e^{-t}\cos t - e^{-t}\sin t$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Partial fraction

$$\frac{1+3s-s^2}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Clear fractions

$$1+3s-s^2 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$1+3s-s^2 = (A+B)s^2 + (-2A-B+C) + A$$

$A + B = -1, \quad -2A - B + C = 3, \quad A = 1$

$$B = -1 - A = -2$$

$$C = 3 + B + 2A = 3 - 2 + 2 = 3$$

So $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s} - \dfrac{2}{s-1} + \dfrac{3}{(s-1)^2}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\,\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\,\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

form is $\dfrac{1}{s^2}$

$$= 1 - 2e^t + 3te^t$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

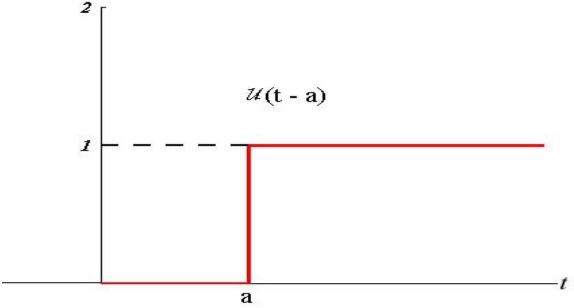

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \quad = \quad g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Consider $0 \le t < a$, then $\mathscr{U}(t-a) = 0$

$f(t) = g(t) - g(t) \cdot 0 + h(t) \cdot 0 = g(t)$ as required

Consider $t \geq a$, then $\mathscr{U}(t-a) = 1$

$f(t) = g(t) - g(t) \cdot 1 + h(t) \cdot 1 = h(t)$ again as required.

So it is true that

$$\begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$$

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

$$f(t) = e^t - e^t\,\mathscr{U}(t-2) + t^2\,\mathscr{U}(t-2) - t^2\,\mathscr{U}(t-5) + 2t\,\mathscr{U}(t-5)$$

turn on

turn off @ $t=2$

turn on @ $t=2$

turn off @ $t=5$

turn on @ $t=5$

Let's verify

For $0 \le t < 2$ $\mathcal{U}(t-2)=0$ and $\mathcal{U}(t-5)=0$

$$f(t)=e^t-e^t\cdot 0+t^2\cdot 0-t^2\cdot 0+2t\cdot 0 = e^t$$

For $2 \le t < 5$, $\mathcal{U}(t-2)=1$ $\mathcal{U}(t-5)=0$

$$f(t)=e^t-e^t\cdot 1+t^2\cdot 1-t^2\cdot 0+2t\cdot 0 = t^2$$

For $t \geq 5$, $\mathcal{U}(t-2)=1$ $\mathcal{U}(t-5)=1$

$$f(t)=e^t-e^t\cdot 1+t^2\cdot 1-t^2\cdot 1+2t\cdot 1 = 2t$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

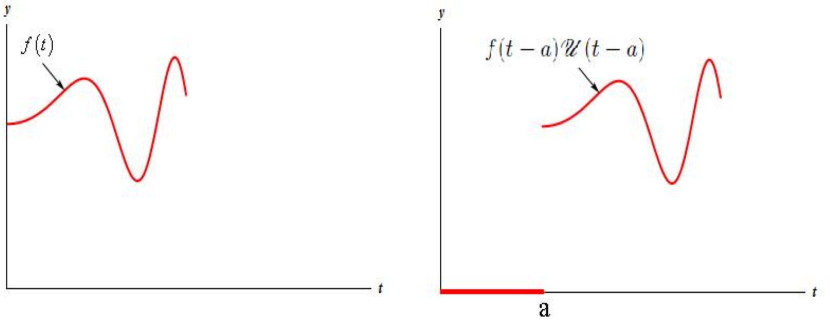

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

Show that $\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}$

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \geq a \end{cases}$$

By definition

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

for $s > 0$

$$= \frac{e^{-st}}{-s}\bigg|_a^\infty = \frac{-1}{s}\left(0 - e^{-s\cdot a}\right)$$

$$= \frac{e^{-as}}{s}$$

Example

Find the Laplace transform $\mathscr{L}\{h(t)\}$ where

$$h(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write h in terms of unit step functions.

$$h(t) = \underline{1} - 1\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + (t-1)\mathcal{U}(t-1)$$

Example Continued...

(b) Now use the fact that $h(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{h\}$.

$$\mathscr{L}\{h(t)\} = \mathscr{L}\{1 + (t-1)\mathscr{U}(t-1)\}$$

$$= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$$

$f(t-a)\mathscr{U}(t-a)$

If $f(t) = t$

then

$f(t-1) = t-1$

$\mathscr{L}\{t\} = \frac{1}{s^2}$

$$= \frac{1}{s} + \frac{1}{s^2}e^{-1s}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2}$$

A Couple of Useful Results

Another formulation of this translation theorem is

(1) $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$

Example: Find $\mathscr{L}\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\} = e^{-\frac{\pi}{2}s}\mathscr{L}\{\cos\left(t+\frac{\pi}{2}\right)\}$

$= e^{-\frac{\pi}{2}s}\mathscr{L}\{-\sin t\}$

$= -e^{-\frac{\pi}{2}s}\mathscr{L}\{\sin t\} = \dfrac{-e^{-\pi/2\, s}}{s^2+1}$

$\cos(t+\pi/2) = \cos t\cos\pi/2 - \sin t\sin\pi/2 = -\sin t$

A Couple of Useful Results

The inverse form of this translation theorem is

$$(2) \quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$

We need $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$

Partial fraction $\dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1}$

$1 = A(s+1) + Bs$

Set $s=0$ $\quad 1 = A$

$s=-1$ $\quad 1 = -B$

So $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$$= \underbrace{1 - e^{-t}}_{\text{this is our } f(t)}$$

So $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2)$

$f(t-2)\,\mathcal{U}(t-2)$