July 14 Math 2254 sec 001 Summer 2015

Section 8.7: Summary of Tests for Series

★ Does it have a specific *type*? (p-series, geometric, telescoping, alternating)

★ If you can readily see that $\lim_{n\to\infty} a_n \neq 0$, use the Divergence test.

★ If $a_n > 0$ and the function $f(n) = a_n$ looks like you can integrate it (i.e. $\int_1^\infty f(x)\,dx$ is manageable), try the integral test.

★ If it involves a rational function in n or a ratio of roots and powers of n, a direct or limit comparison test (comparing to a p-series) might be useful.

★ If it looks very similar to a geometric series, but is not quite a geometric series, a direct or limit comparison test to a geometric may be useful.

★ If it involves factorials or complicated products, the ratio test might lead to the necessary conclusion. If it involves expressions to the n^{th} power, the root test may work.

◇ Remember that the ratio & root tests (when conclusive) determine absolute convergence. When using the alternating series test, if a series is found to be convergent remember to check for absolute convergence.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{3n+2}{n-\sqrt{2}}$

Divergence test

$$\lim_{n\to\infty} \frac{3n+2}{n-\sqrt{2}} = \lim_{n\to\infty} \left(\frac{3n+2}{n-\sqrt{2}}\right) \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{3+\frac{2}{n}}{1-\frac{\sqrt{2}}{n}} = \frac{3+0}{1-0} = 3 \neq 0$$

The series diverges by the divergence test.

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{3n}{2n^2+3}$

Alt. Series test: $a_n = \dfrac{3n}{2n^2+3}$

i) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{2n^2+3}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}}$

$\displaystyle = \lim_{n\to\infty}\frac{\frac{3}{n}}{2+\frac{3}{n^2}} = \frac{0}{2+0} = 0$

ii) Is $a_{n+1} \le a_n$? Let $f(x) = \dfrac{3x}{2x^2+3}$

$$f'(x) = \frac{3(2x^2+3)-3x(4x)}{(2x^2+3)^2} = \frac{6x^2+9-12x^2}{(2x^2+3)^2} = \frac{9-6x^2}{(2x^2+3)^2}$$

If $9-6x^2<0 \implies \frac{9}{6}<x^2 \implies \sqrt{\frac{3}{2}}<x$

So $f'(x)<0$ for $x \geqslant 2$

$a_{n+1} = f(n+1) < f(n) = a_n$ for $n \geqslant 2$.

Both conditions of the alt. Series test are true. So the Series is convergent.

To determine the type of convergence, consider

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{3n}{2n^2+3} \right| = \sum_{n=1}^{\infty} \frac{3n}{2n^2+3}$$

As $n \to \infty$ $\quad \dfrac{3n}{2n^2+3} \sim \dfrac{3n}{2n^2} = \dfrac{3}{2}\dfrac{1}{n}$

Use limit comparison with the divergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Let $a_n = \dfrac{3n}{2n^2+3}$ and $b_n = \dfrac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n}{2n^2+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^2}{2n^2+3} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{3}{2+\frac{3}{n^2}} = \frac{3}{2+0} = \frac{3}{2}$$

$0 < \frac{3}{2} < \infty$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the

Series $\sum_{n=1}^{\infty} \frac{3n}{2n^2+3}$ also diverges.

The series $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{2n^2+3}$ is

<u>Conditionally</u> <u>convergent</u>.

Section 8.8: Power Series

Motivating Example: Let x be a variable (representing a real number). Show that the series

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n^2}$$

converges if $x = 3$ and diverges if $x = 7$.

Let $x=3$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(3-4)^n}{2n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2}$

Note $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{2n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ a convergent p-series.

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2}$ is absolutely convergent.

Setting $x=7$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(7-4)^n}{2n^2}=\sum_{n=1}^{\infty}\frac{3^n}{2n^2}$

Ratio test w/ $a_n=\dfrac{3^n}{2n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{3^{\cancel{n}+1}}{\cancel{2}(n+1)^2}\cdot\frac{\cancel{2}n^2}{\cancel{3^n}}\right|$$

$$=\lim_{n\to\infty}\frac{3n^2}{(n+1)^2}=\lim_{n\to\infty}3\left(\frac{n}{n+1}\right)^2$$

$$=\lim_{n\to\infty}3\left(\frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right)^2$$

$$= \lim_{n\to\infty} 3\left(\frac{1}{1+\frac{1}{n}}\right)^2 = 3(1)^2 = 3$$

$L = 3 > 1$ so the series diverges when $x = 7$.

Power Series

Definition: A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

where the a_n's are (known) constants called the **coefficients**, x is a variable, and c is a (known) constant called the **center**.

For convenience, we set $(x-c)^0 = 1$ even in the case that $x = c$.

Remark: As the previous example suggests, a power series may be convergent for some values of x and divergent for others.

Example

Determine all value(s) of x for which the series converges.

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n^2}$$

If $x=4$, the series terms are all zero.

Ratio test: $a_n = \dfrac{(x-4)^n}{2n^2}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-4)^{\cancel{n+1}}}{\cancel{2}(n+1)^2} \cdot \frac{\cancel{2}n^2}{\cancel{(x-4)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-4)\, n^2}{(n+1)^2} \right|$$

$$= \lim_{n\to\infty} |x-4| \left(\frac{n}{n+1}\right)^2 = |x-4|(1)^2$$

$$= |x-4|$$

$L = |x-4|$ The series converges absolutely

if $L < 1 \Rightarrow |x-4| < 1$

i.e. $-1 < x - 4 < 1$

$+4 \quad +4 \quad +4$

$$3 < x < 5$$

$L=1$ if $x=3$ or $x=5$

We know it converges if $x=3$.

If $x=5$ the series is $\sum_{n=1}^{\infty} \frac{(5-4)^n}{2n^2} = \sum_{n=1}^{\infty} \frac{1}{2n^2}$

which is a convergent p-series,

$$\boxed{\text{The series converges absolutely if } 3 \leq x \leq 5}$$

If $L > 1$, i.e. $|x-4| > 1$

The series diverges by the ratio test.

So the series diverges if $x > 5$ or $x < 3$.