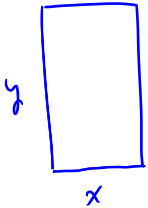
July 17 Math 1190 sec. 51 Summer 2017

Section 4.7: Optimization

Show that among all rectangles with perimeter $\mathscr{P}$, the one with the largest area is a square.



representative rectangle

call the length and width y and x.

The area $A = xy$. The perimeter is

$P = 2x + 2y$.

Maximize A given $P = \mathscr{P}$

$$\mathscr{P} = 2x + 2y \implies y = \frac{\mathscr{P} - 2x}{2} = \frac{\mathscr{P}}{2} - x$$

So $A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$

Find critical #s: $A'(x) = \frac{P}{2} - 2x$

$A'(x)$ is always defined,

$A'(x) = 0 \implies \frac{P}{2} - 2x = 0 \implies x = \frac{P}{4}$

Verify we have a max: 2nd der. test

$A''(x) = -2 \implies A''\left(\frac{P}{4}\right) = -2 < 0$

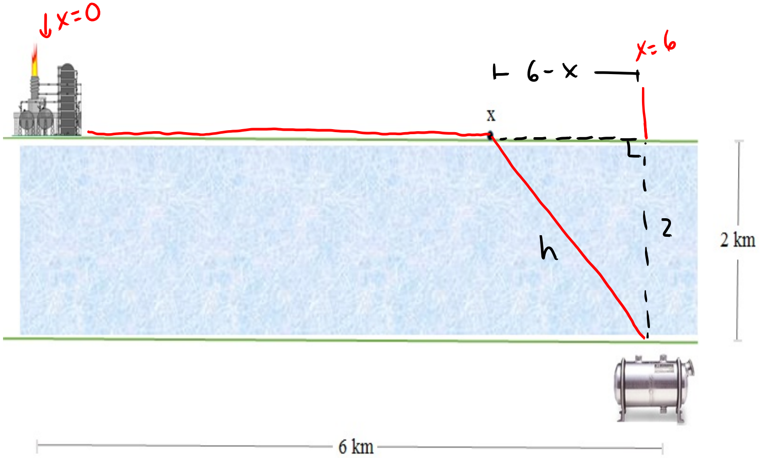
So $x = \frac{P}{4}$ maximizes A.

If $x = \frac{P}{4}$, then $y = \frac{P}{2} - \frac{P}{4} = \frac{P}{4} = x$

So the rectangle with the largest area has equal length and width making it a square.

Applied Optimization Example

An oil refinery is located on the north bank of a straight river that is 2 km wide. Storage tankes are located on the south bank of the river and 6 km east of the refinery. (see the figure)



Oil refinery problem continued...

It costs \$ 400,000 per km to lay pipe over land to a point x on the north bank, and it costs \$ 800,000 per km to lay pipe under water to reach the tanks on the south bank. Where should the point x be located to minimize the cost of the pipeline?

x could be 6km (2km underwater)

x could be 0km (all pipe underwater)

or somewhere in between.

We need a cost function.

Cost = Cost underground + cost underwater

Taking cost in 10^5 \$

$$C(x) = 4x + 8h$$

where h is the length of pipe underwater.

$$h^2 = 2^2 + (6-x)^2 \Rightarrow h = \sqrt{(6-x)^2 + 4}$$

So
$$C(x) = 4x + 8\sqrt{(6-x)^2 + 4}$$

Find critical #s :

$$c'(x) = 4 + 8\left(\cancel{\tfrac{1}{2}}\right)\left((x-6)^2+4\right)^{-1/2} \cdot \cancel{2}(6-x)\cdot(-1)$$

$$c'(x) = 4 - \frac{8(6-x)}{\sqrt{(6-x)^2+4}}$$

$c'(x)$ is always defined

$$c'(x) = 0 \Rightarrow 4 - \frac{8(6-x)}{\sqrt{(6-x)^2+4}} = 0$$

$$4 = \frac{8(6-x)}{\sqrt{(6-x)^2+4}}$$

$$\sqrt{(6-x)^2+4} = 2(6-x)$$

square

$$\left(\sqrt{(6-x)^2+4}\right)^2 = \left(2(6-x)\right)^2$$

$$(6-x)^2 + 4 = 4(6-x)^2$$

$$4 = 3(6-x)^2$$

$$(6-x)^2 = \frac{4}{3}$$

$$6-x = \pm\sqrt{\frac{4}{3}} = \frac{\pm 2}{\sqrt{3}}$$

$$x = 6 \mp \frac{2}{\sqrt{3}}$$

i.e. $\quad x = 6 - \frac{2}{\sqrt{3}} \quad$ or $\quad x = 6 + \frac{2}{\sqrt{3}}$

We have one critical number in the interval $0 \le x \le 6$.

We can check the cost if $x=0$, $x=6$ or

$x = 6 - \frac{2}{\sqrt{3}}$.

$$C(x) = 4x + 8\sqrt{(6-x)^2+4}$$

$$C(0) = 4\cdot 0 + 8\sqrt{(6-0)^2+4} = 8\sqrt{36+4} = 8\sqrt{40} = 16\sqrt{10}$$

$$C(6) = 4\cdot 6 + 8\sqrt{(6-6)^2+4} = 24 + 8\cdot 2 = 40$$

$$C\left(6-\frac{2}{\sqrt{3}}\right) = 4\left(6-\frac{2}{\sqrt{3}}\right) + 8\sqrt{\left(6-6+\frac{2}{\sqrt{3}}\right)^2+4}$$

$$= 24 - \frac{8}{\sqrt{3}} + 8\sqrt{\frac{4}{3}+4}$$

$$= 24 - \frac{8}{\sqrt{3}} + 8\sqrt{\frac{16}{3}} = 24 - \frac{8}{\sqrt{3}} + \frac{32}{\sqrt{3}}$$

$$C\left(6 - \frac{2}{\sqrt{3}}\right) = 24 + \frac{24}{\sqrt{3}} = 24 + 8\sqrt{3}$$

The three costs are

$$C(0) = 16\sqrt{10}$$

$$C\left(6 - \frac{2}{\sqrt{3}}\right) = 24 + 8\sqrt{3} \leftarrow \text{smallest}$$

$$C(6) = 40$$

The optimal position for x is $6-\frac{2}{\sqrt{3}} \approx 4.85$ km east of the refinery.

Calculator Results

For $C(x) = 4x + 8\sqrt{(6-x)^2 + 2^2}$,

$$C(0) = 16\sqrt{10} \approx 50.6$$

$$C\left(6 - \frac{2}{\sqrt{3}}\right) = 24+8\sqrt{3} \approx 37.9$$

$$C(6) = 40$$

$$6 - \frac{2}{\sqrt{3}} \approx 4.85$$

Question

The potential drop (voltage) V across a capacitor is

$$V = \frac{Q}{C}$$

where Q is the charge on the capacitor and C is the capacitance. If V, Q, and C are all functions of time t, then relate $\frac{dV}{dt}$ to $\frac{dQ}{dt}$ and $\frac{dC}{dt}$.

(a) $$\frac{dV}{dt} = \frac{\frac{dQ}{dt}}{\frac{dC}{dt}}$$

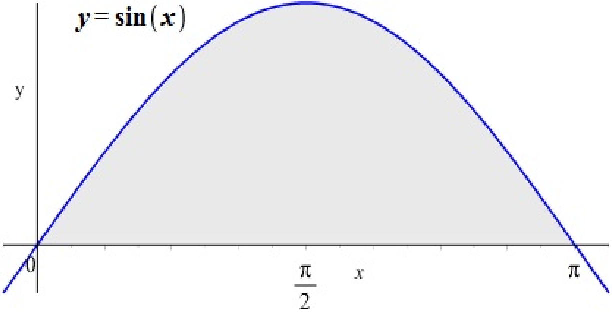
(b) $$\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt} + \frac{Q}{\frac{dC}{dt}}$$

(c) $$\frac{dV}{dt} = \frac{\frac{dQ}{dt}C - \frac{dC}{dt}Q}{C^2}$$

Question

The area of the shaded region is

$\text{Area} = \int_0^{\pi} \sin(x)\,dx$



(a) 1

(b) π

(c) 2π

(d) 2

Example

Let $f(x) = e^{-x^2}$ and let $g(x) = \displaystyle\int_1^x f(t)\,dt$.

Find the equation of the line tangent to the graph of g at the point $(1, g(1))$.

The slope $m = g'(1)$.

$$g'(x) = \frac{d}{dx}\left[\int_1^x f(t)\,dt\right] = f(x) = e^{-x^2}$$

$$g'(1) = e^{-1^2} = e^{-1} \qquad g(1) = \int_1^1 f(t)\,dt = 0$$

pt $(1,0)$ slope $m = e^{-1}$ $\qquad y - 0 = e^{-1}(x-1) \Rightarrow \boxed{y = e^{-1}x - e^{-1}}$

Question

$$\text{If} \quad g(x) = \int_1^x e^{-t^2}\, dt \quad \text{then } g''(x) =$$

(a) e^{-x^2}

(b) $-2xe^{-x^2}$

(c) $e^{-x^2} - e^{-1}$

(d) $-e^{-x^2}$

$g'(x) = e^{-x^2}$

$g''(x) = e^{-x^2}(-2x) = -2x e^{-x^2}$

Example

Determine the intervals over which g is increasing and the intervals over which it's decreasing

$$\text{where} \quad g(x) = \int_1^x e^{-t^2}\,dt$$

$g'(x) = e^{-x^2}$

$g'(x)$ is always defined

$g'(x) = 0 \Rightarrow 0 = e^{-x^2}$ no solutions.

$g'(x) > 0$ for all real x

g is increasing on $(-\infty, \infty)$.

Question

$$\text{If} \quad g(x) = \int_1^x e^{-t^2}\,dt, \quad \text{then the concavity of } g \text{ is}$$

$g''(x) = -2x\,e^{-x^2}$

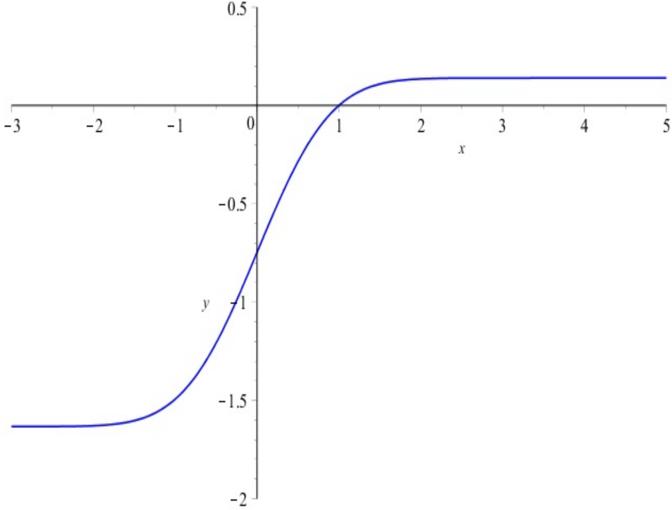
(a) g is everywhere concave up

(b) g is everywhere concave down

(c) g is concave up if $x < 0$ and concave down if $x > 0$

(d) g is concave down if $x < 0$ and concave up if $x > 0$

The graph of $g(x) = \int_1^x e^{-t^2}\,dt$



Average Value of a Function

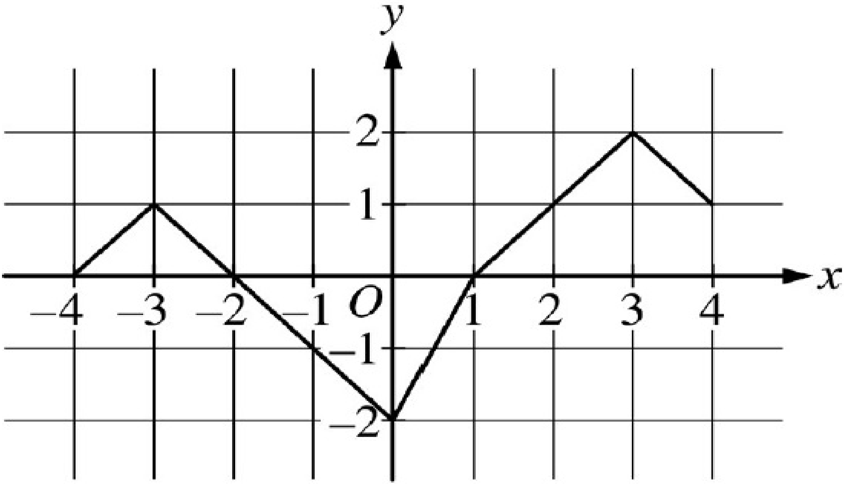
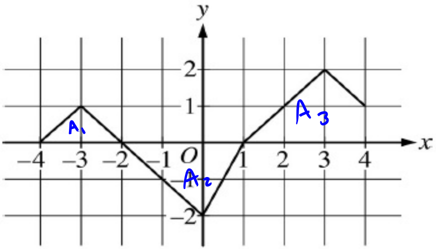


Figure: Find the average value of f on $[-4, 4]$.

Question



What is the average value of f on the interval $-4 \le x \le 4$?

(A) $\frac{1}{8}$

(B) $\frac{3}{16}$

(C) $\frac{15}{16}$

(D) $\frac{3}{2}$

$A_1 = 1$ $\quad A_3 = 3.5$

$A_2 = 3$

$$\int_{-4}^{4} f(x)\,dx = 1 + 3.5 - 3 = \frac{3}{2}$$

$$f_{avg} = \frac{1}{4-(-4)} \int_{-4}^{4} f(x)\,dx = \frac{1}{8} \cdot \frac{3}{2} = \frac{3}{16}$$