July 18 Math 2306 sec 52 Summer 2016

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform and that f is differentiable on $[0,\infty)$. Obtain an expression for the Laplace tranform of $f'(t)$. (Assume f is of exponential order c for some c.)

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Int. by parts

$u = e^{-st}$ $\quad du = -se^{-st}\,dt$

$v = f(t)$ $\quad dv = f'(t)\,dt$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\,dt$$

$$= 0 - e^0 f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$= -f(0) + s\mathcal{L}\{f(t)\}$

If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

Determine $\mathscr{L}\{f''(t)\}$ in terms of $\mathscr{L}\{f(t)\}$.

$$\mathscr{L}\{f''(t)\} = s\,\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\,\mathscr{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2 F(s) - s\,f(0) - f'(0)$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{g(t)\} = G(s)$

Take transform of the ODE

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{g(t)\}$$

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = G(s)$$

$$a s^2 Y(s) - a s y(0) - a y'(0) + b s Y(s) - b y(0) + c Y(s) = G(s)$$

$$(a s^2 + b s + c) Y(s) - a y_0 s - a y_1 - b y_0 = G(s)$$

Let's isolate $Y(s)$

$$\underbrace{(a s^2 + b s + c)}_{\text{Characteristic polynomial for the ODE}} Y(s) = a y_0 s + a y_1 + b y_0 + G(s)$$

$$Y(s) = \frac{ay_0 s + ay_1 + by_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

With $Y(s)$ known, we can find the solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Solving IVPs

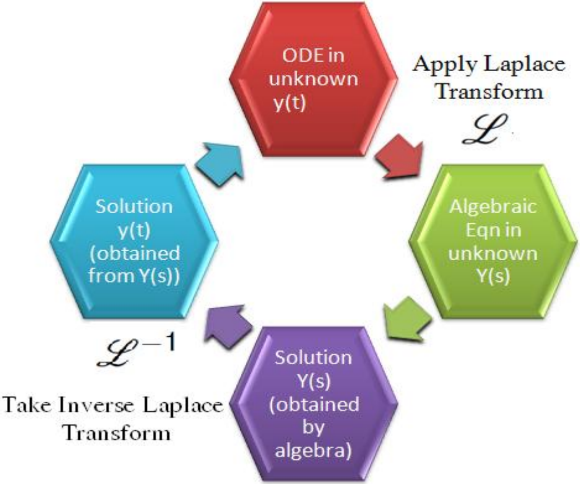

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**

Solve the IVP using the Laplace Transform

$$\frac{dy}{dt}+3y=2t \quad y(0)=2$$

Take transform of the ODE

Let $\mathscr{L}\{y\} = Y(s)$

$$\mathscr{L}\{y' + 3y\} = \mathscr{L}\{2t\}$$

$$\mathscr{L}\{y'\} + 3\mathscr{L}\{y\} = 2\mathscr{L}\{t\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{2}{s^2}$$

$$(s+3)Y(s) - 2 = \frac{2}{s^2}$$

$$(s+3)Y(s) = 2 + \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s+3} + \frac{2}{s^2(s+3)}$$

We need to find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$. Do a partial frac.

Decomp on

$$\frac{2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$\Longrightarrow \quad 2 = As(s+3) + B(s+3) + Cs^2$$

$$= A(s^2+3s) + B(s+3) + Cs^2$$

$$0s^2 + 0s + 2 = (A+C)s^2 + (3A+B)s + 3B$$

$$A + C = 0 \qquad \Rightarrow C = -A = \frac{2}{9}$$

$$3A + B = 0 \qquad \Rightarrow A = \frac{-1}{3}B = \frac{-2}{9}$$

$$3B = 2 \Rightarrow B = \frac{2}{3}$$

$$Y(s) = \frac{2}{s+3} + \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{2/9}{s+3}$$

$$Y(s) = \frac{20/9}{s+3} - \frac{2/9}{s} + \frac{2/3}{s^2}$$

Find y by taking the inverse transform.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{20/9}{s+3} - \frac{2/9}{s} + \frac{2/3}{s^2}\right\}$$

$$= \frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\boxed{y(t) = \frac{20}{9}e^{-3t} - \frac{2}{9} + \frac{2}{3}t}$$

Solve the IVP using the Laplace Transform

$$y'' + 4y' + 4y = te^{-2t} \quad y(0) = 1, y'(0) = 0$$

$\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{te^{-2t}\}$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{te^{-2t}\}$$

$\leftarrow$ shifting theorem

$$s^2Y(s) - sy(0) - y'(0) + 4\left(sY(s) - y(0)\right) + 4Y(s) = \frac{1}{(s+2)^2}$$

$$(s^2+4s+4)Y(s) - s\cdot 1 - 4\cdot 1 = \frac{1}{(s+2)^2}$$

$$(s^2+4s+4)\,Y(s) = s+4 + \frac{1}{(s+2)^2}$$

$$Y(s) = \frac{s+4}{s^2+4s+4} + \frac{1}{(s+2)^2(s^2+4s+4)}$$

Note $\quad s^2+4s+4 = (s+2)^2$

$$Y(s) = \frac{s+4}{(s+2)^2} + \frac{1}{(s+2)^2(s+2)^2}$$

$$= \frac{s+4}{(s+2)^2} + \frac{1}{(s+2)^4}$$

Use $\dfrac{s+4}{(s+2)^2} = \dfrac{s+2+2}{(s+2)^2} = \dfrac{s+2}{(s+2)^2} + \dfrac{2}{(s+2)^2} = \dfrac{1}{s+2} + \dfrac{2}{(s+2)^2}$

So $\Upsilon(s) = \dfrac{1}{s+2} + \dfrac{2}{(s+2)^2} + \dfrac{1}{(s+2)^4}$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 2t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3!}\,\frac{3!}{s^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{6}t^3$$

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

$$= e^{-2t} + 2te^{-2t} + \frac{1}{6}t^3e^{-2t}$$

The soln. to the IVP is

$$y(t) = e^{-2t} + 2te^{-2t} + \frac{1}{6}t^3e^{-2t}$$