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Section 8.8: Power Series

Definition: A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

where the a_n's are (known) constants called the **coefficients**, x is a variable, and c is a (known) constant called the **center.**

For convenience, we set $(x-c)^0 = 1$ even in the case that $x = c$.

The power series converges when $x = c$. In this case, the series is equal to a_0.

Example

Determine all value(s) of x for which the series converges.

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n^2}$$

We found, using the **ratio test**, that this power series will converge if $3 \leq x \leq 5$ and will diverge if $x > 5$ or if $x < 3$. We might note here the the set of x values for which the series converges is an interval. Moreover, the center $c = 4$ happens to be the exact midpoint of that interval.

Example

Determine all value(s) of x for which the series converges.

$$\sum_{n=0}^{\infty} n!x^n$$

This converges at its center $c = 0$.

Ratio test: $a_n = n!\,x^n$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\,\cancel{x}^{n+1}}{n!\,\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}\,x}{\cancel{n!}} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

Here $L = \infty$ for all $x \neq 0$.

$L > 1$ for all $x \neq 0$, so this series doesn't converge for any x other than the center.

Example

Determine all value(s) of x for which the series converges.

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

The center is $c = 0$.

Ratio test: $a_n = (-1)^n \dfrac{x^{2n}}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n}+1}x^{\cancel{2n}+2}}{(2n+2)!}\cdot\frac{(2n)!}{\cancel{(-1)^n}\,\cancel{x^{2n}}}\right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\, x^2 \, \cancel{(2n)!}}{\cancel{(2n)!}(2n+1)(2n+2)} \right|$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+1)(2n+2)} = 0$$

Here $L = 0 < 1$ for all real x.

The series converges absolutely for all real x.

Theorem on Power Series Convergence

Theorem: For the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, there are three possibilities:

(i) The series converges at the center $x = c$ and nowhere else.

(ii) The series converges for all real x; or

(iii) There exists a positive number R such that the series converges if $|x-c| < R$ and diverges if $|x-c| > R$.

In the third case, R is called the **radius of convergence.**

Case (iii): Interval of Convergence

If there is a finite radius of convergence R, then the series converges for $|x - c| < R$. That is, for

$$c - R < x < c + R.$$

Behavior at the end points $x = c - R$ or $x = c + R$ varies from series to series. There are four possible cases. The **interval of convergence** may be any one of the following:

$$(i)\ c - R < x < c + R, \qquad (ii)\ c - R \le x < c + R,$$
$$(iiii)\ c - R < x \le c + R, \quad \text{or} \quad (iv)\ c - R \le x \le c + R.$$

Example

Determine the radius and interval of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{n(x+1)^n}{4^n}$$

The center for this one is $c=-1$.

Ratio test: $a_n = \dfrac{n(x+1)^n}{4^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)(x+1)^{\cancel{n+1}}}{4^{\cancel{n+1}}} \cdot \frac{\cancel{4^n}}{n\cancel{(x+1)^n}} \right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)(x+1)}{4n}\right| = \lim_{n\to\infty} \frac{1}{4}|x+1| \left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{4}|x+1|\left(1+\frac{1}{n}\right) = \frac{|x+1|}{4}(1)$$

Here $L = \frac{|x+1|}{4}$. The series converges absolutely if $L<1$ — i.e.

$$\frac{|x+1|}{4} < 1 \implies |x+1| < 4$$

The radius of convergence is 4.

$$|x+1| < 4 \implies -4 < x+1 < 4$$

$$-5 < x < 3$$

End point check:

$x = 3$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{n(3+1)^n}{4^n} = \sum_{n=1}^{\infty} \frac{n\,4^n}{4^n} = \sum_{n=1}^{\infty} n$

Divergence test $\quad \displaystyle\lim_{n\to\infty} n = \infty \neq 0$

The series diverges when $x = 3$.

$x = -5$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{n(-5+1)^n}{4^n} = \sum_{n=1}^{\infty} \frac{n(-4)^n}{4^n}$

$$= \sum_{n=1}^{\infty} n\left(\frac{-4}{4}\right)^n = \sum_{n=1}^{\infty} n(-1)^n$$

Divergence test $\lim_{n\to\infty} n(-1)^n$ DNE

The series diverges when $x = -5$

The radius and interval of convergence are

$R = 4$ and $I = (-5, 3)$

Example

Determine the radius and interval of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{2^n x^n}{\sqrt{n}}$$

Ratio test: $a_n = \dfrac{2^n x^n}{\sqrt{n}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{\cancel{n}+1} x^{\cancel{n}+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\cancel{2^n} \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2x\sqrt{n}}{\sqrt{n+1}} \right| = \lim_{n\to\infty} 2|x| \sqrt{\frac{n}{n+1}}$$

$$= \lim_{n\to\infty} 2|x| \sqrt{\frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}} = \lim_{n\to\infty} 2|x|\sqrt{\frac{1}{1+\frac{1}{n}}}$$

$$= 2|x|\sqrt{1} = 2|x| \qquad L = 2|x|.$$

The series converges absolutely if $(L < 1)$

$$2|x| < 1 \Rightarrow |x| < \frac{1}{2}$$

The radius $R = \frac{1}{2}$.

$$|x| < \frac{1}{2} \implies -\frac{1}{2} < x < \frac{1}{2}$$

End point check:

$x = \frac{1}{2}$ $$\sum_{n=1}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\left(\frac{2}{2}\right)^n}{\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$ p-series w/ $p = \frac{1}{2}$

divergent.

The series diverges if $x = \frac{1}{2}$

$x = \frac{-1}{2}$ $\quad \sum_{n=1}^{\infty} \frac{2^n \left(\frac{-1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\left(\frac{-2}{2}\right)^n}{\sqrt{n}}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

alt. series test (i) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

(ii) $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ The series converges.

The series converges (conditionally) @ $x = \frac{-1}{2}$.

The radius $R = \frac{1}{2}$, the interval is $I = \left[\frac{-1}{2}, \frac{1}{2}\right)$

Example

Determine the radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

Ratio test: $a_n = \dfrac{x^n}{1\cdot 3\cdot 5\cdots(2n+1)}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{\cancel{n}+1}}{1\cdot 3\cdot 5\cdots(2(n+1)+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2n+1)}{\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x\,(\cancel{1\cdot 3\cdot 5\cdots(2n+1)})}{\cancel{1\cdot 3\cdot 5\cdots}\,(2n+3)} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{2n+3} = 0 \qquad L = 0 < 1$$

for all real x.

The series converges absolutely for all real x.

The radius $R = \infty$ and the interval

$$I = (-\infty, \infty)$$

$$\sum_{n=0}^{\infty} \frac{x^n}{1\cdot 3\cdot 5\cdots(2n+1)} = \frac{x^0}{1} + \frac{x^1}{1\cdot 3} + \frac{x^2}{1\cdot 3\cdot 5} +$$

$$+ \frac{x^3}{1\cdot 3\cdot 5\cdot 7} + \frac{x^4}{1\cdot 3\cdot 5\cdot 7\cdot 9} + \cdots$$

Functions as Power Series

Motivating Example: Let

$$f(x) = \frac{1}{1-x}, \quad \text{for} \quad -1 < x < 1.$$

Use the well known relation $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$ to express f as a power series.

$$\frac{1}{1-x} = \frac{a}{1-r} \quad \text{if} \quad a=1 \quad \text{and} \quad r=x$$

$$\text{for} \quad -1<x<1 \qquad |r| = |x| < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} 1x^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

Using Part of a Series to Approximate f

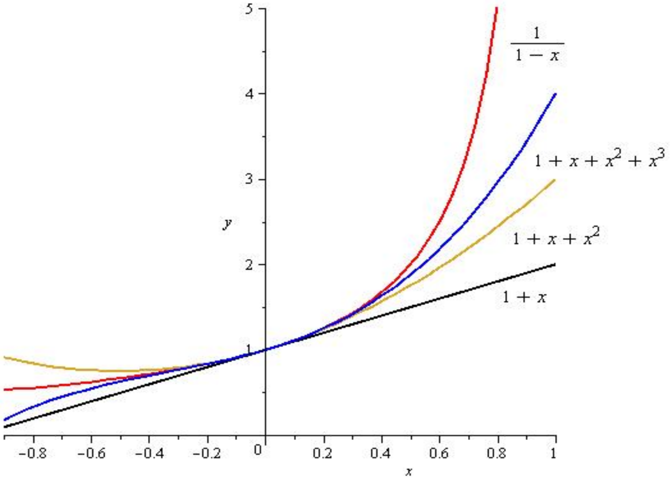

Figure: Plot of f along with the first 2, 3, and 4 terms of the series. Near the center, the graphs agree well. The fit breaks down away from the center.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ for } |r| < 1$$

Find a power series representation, in powers of x, of the rational function. Indicate the interval of convergence.

$$f(x) = \frac{1}{1+x^2}$$

$\frac{1}{1+x^2} = \frac{a}{1-r}$ if $a=1$ and $r = -x^2$

If $|r| = |-x^2| < 1 \Rightarrow |x^2| < 1 \Rightarrow x^2 < 1$

i.e. $-1 < x < 1$

for $-1 < x < 1$ $\quad f(x) = \sum_{n=0}^{\infty} 1(-x^2)^n$

$$f(x) = \sum_{n=0}^{\infty} (-1 \cdot x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \, x^{2n}$$

$$f(x) = 1 - x^2 + x^4 - x^6 + \ldots$$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ for } |r| < 1$$

the 1 in this formula ~~must~~ must be here.

Find a power series representation, in powers of x, of the rational function. Indicate the interval of convergence.

$$f(x) = \frac{1}{x-3} = \frac{-1}{3-x} = \frac{-1}{3\left(1-\frac{x}{3}\right)} = \frac{\frac{-1}{3}}{1-\frac{x}{3}}$$

$$\frac{\frac{-1}{3}}{1-\frac{x}{3}} = \frac{a}{1-r} \quad \text{if} \quad a = \frac{-1}{3} \quad \text{and} \quad r = \frac{x}{3}$$

$$|r| = \left|\frac{x}{3}\right| < 1 \Rightarrow \frac{|x|}{3} < 1 \Rightarrow |x| < 3$$

i.e. $-3 < x < 3$

For $-3 < x < 3$

$$f(x) = \sum_{n=0}^{\infty} \frac{-1}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{-1}{3}\,\frac{x^n}{3^n}$$

$$= -\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$$

Theorem: Differentiation and Integration

Theorem: Let $\sum a_n(x-c)^n$ have positive radius of convergence R, and let the function f be defined by this power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Then f is differentiable on $(c-R, c+R)$. Moreover,

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}.$$

Theorem Continued

Moreover, f can be integrated term by term

$$\begin{aligned}\int f(x)\,dx &= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots \\ &= C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}\end{aligned}$$

The radius of convergence for each of these series is R.

Use Differentiation to *Guess* a Function

Let $f(x)$ be given by the following power series. Take at least one derivative, and see if you can guess exactly what function f is.

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Note $f(0) = 1 + 0 + \frac{0^2}{2!} + \cdots = 1$

$$\boxed{f(x) = e^x}$$

$$f'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \cdots$$

$$= 1 + \frac{2x}{1\cdot 2} + \frac{3x^2}{1\cdot 2\cdot 3} + \frac{4x^3}{1\cdot 2\cdot 3\cdot 4} + \frac{5x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = f(x)$$

$f'(x) = f(x)$