July 20 Math 2306 sec 52 Summer 2016

Section 16: Laplace Transforms of Derivatives and IVPs

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Solving an IVP with Laplace Transforms

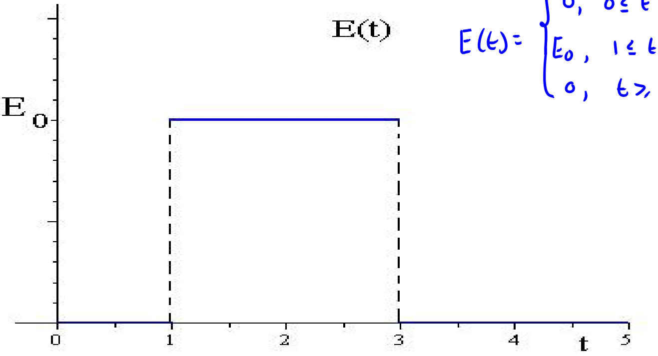
We'll solve the constant coefficient IVP $ay'' + by' + cy = g(t)$ subject to $y(0) = y_0$, $y'(0) = y_1$ by

- taking the Laplace transform of both sides of the ODE using properties and the table;
- inserting the initial conditions, and solving algebraically for $Y(s) = \mathscr{L}\{y(t)\}$;
- expanding $Y(s)$ if needed with a partial fraction decomposition; and
- using the table to find $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.

$L\frac{di}{dt} + Ri = E, \quad i(0) = 0$



$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \geq 3 \end{cases}$$

LR Circuit Example

$$\frac{di}{dt} + 10i = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}, \quad i(0) = 0$$

$$= 0 - 0\,\mathcal{U}(t-1) + E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3)$$

$$\frac{di}{dt} + 10i = E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3)$$

Let $I(s) = \mathcal{L}\{i(t)\}$

$$\mathcal{L}\{i'\} + 10\mathcal{L}\{i\} = E_0\,\mathcal{L}\{\mathcal{U}(t-1)\} - E_0\,\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$sI(s) - \underbrace{i(0)}_{=0} + 10I(s) = \frac{E_0}{s}e^{-s} - \frac{E_0}{s}e^{-3s}$$

$$(s+10)I(s) = \frac{E_0}{s}e^{-s} - \frac{E_0}{s}e^{-3s}$$

$$I(s) = \frac{E_0}{s(s+10)}e^{-s} - \frac{E_0}{s(s+10)}e^{-3s}$$

Partial fractions $\quad \frac{E_0}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$

$$E_0 = A(s+10) + Bs$$

Set $\quad s=0 \quad E_0 = 10A \Rightarrow A = \frac{E_0}{10}$

$\quad s=-10 \quad E_0 = -10B \Rightarrow B = \frac{-E_0}{10}$

$$I(s) = \frac{E_0}{10}\left(\frac{1}{s} - \frac{1}{s+10}\right)e^{-s} - \frac{E_0}{10}\left(\frac{1}{s} - \frac{1}{s+10}\right)e^{-3s}$$

$$i(t) = \mathcal{L}^{-1}\left\{I(s)\right\}$$

$$\boxed{i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)\mathcal{U}(t-1) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)\mathcal{U}(t-3)}$$

This is the current for $t \geq 0$

Written in a traditional piecewise form

$$i(t) = \begin{cases} 0, & 0 \le t < 1 \\ \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right), & 1 \le t < 3 \\ \frac{E_0}{10}\left(e^{-10(t-3)} - e^{-10(t-1)}\right), & t \geq 3 \end{cases}$$

If $0 \le t < 1$ $\quad u(t-1) = 0, \quad u(t-3) = 0$

If $1 \le t < 3$ $\quad u(t-1) = 1, \quad u(t-3) = 0$

If $t \geq 3$ $\quad u(t-1) = 1$ and $u(t-3) = 1$

Solve the IVP using the Laplace Transform

$$y'' + 2y' + 2y = 2t, \quad y(0) = 2, \quad y'(0) = -7$$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 2\mathcal{L}\{t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{2}{s^2}$$

$$s^2Y(s) - 2s + 7 + 2\left(sY(s) - 2\right) + 2Y(s) = \frac{2}{s^2}$$

$$(s^2 + 2s + 2)Y(s) - 2s + 3 = \frac{2}{s^2}$$

$$(s^2+2s+2)Y(s) = \frac{2}{s^2} + 2s - 3$$

$$Y(s) = \frac{2}{s^2(s^2+2s+2)} + \frac{2s-3}{s^2+2s+2}$$

$s^2+2s+2 \qquad b^2-4ac = 2^2-4\cdot1\cdot2 = 4-8 < 0$

Partial fractions

$$\frac{2}{s^2(s^2+2s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+2s+2}$$

$$2 = As(s^2+2s+2) + B(s^2+2s+2) + (Cs+D)s^2$$

$$2 = s^3(A+C) + s^2(2A+B+D) + s(2A+2B) + 2B$$

$$2B = 2 \Rightarrow B = 1$$

$$2A+2B = 0 \Rightarrow A = -B = -1$$

$$2A+B+D = 0 \Rightarrow D = -B-2A = -1-2(-1) = 1$$

$$A+C = 0 \Rightarrow C = -A = 1$$

$$Y(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{s+1}{s^2+2s+2} + \frac{2s-3}{s^2+2s+2}$$

Note $s^2+2s+2 = (s+1)^2+1$ complete the square

$$Y(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{s+1}{(s+1)^2+1} + \frac{2s-3}{(s+1)^2+1}$$

Use $2s-3 = 2(s+1-1)-3 = 2(s+1)-2-3 = 2(s+1)-5$

$$Y(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{s+1}{(s+1)^2+1} + \frac{2(s+1)}{(s+1)^2+1} - \frac{5}{(s+1)^2+1}$$

$$Y(s) = \frac{-1}{s} + \frac{1}{s^2} + 3\,\frac{s+1}{(s+1)^2+1} - 5\,\frac{1}{(s+1)^2+1}$$

$$\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}\ , \quad \mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$y(t) = -1 + t + 3e^{-t}\cos t - 5e^{-t}\sin t$$

Semester Review

Solve the IVP using any applicable method. $y' + 2y = xe^{-x}, \; y(0) = 3$

1st order linear $\quad P(x) = 2$

Integrating factor $\quad \mu = e^{\int P(x)\,dx} = e^{\int 2\,dx} = e^{2x}$

$$\Rightarrow \quad \frac{d}{dx}\left[e^{2x}y\right] = xe^{-x}\cdot e^{2x} = xe^{x}$$

$$\int \frac{d}{dx}\left[e^{2x}y\right]dx = \int xe^{x}\,dx$$

$$e^{2x}y = xe^{x} - \int e^{x}\,dx$$

$$= xe^{x} - e^{x} + C$$

parts

$u = x \qquad du = dx$

$v = e^{x} \qquad dv = e^{x}\,dx$

$$\Rightarrow \quad y = \frac{xe^{x} - e^{x} + C}{e^{2x}} = xe^{-x} - e^{-x} + Ce^{-2x}$$

Apply $y(0) = 3$

$$y(0) = 0e^{0} - e^{0} + Ce^{0} = 3$$

$$-1 + C = 3 \Rightarrow C = 4$$

So $$y = xe^{-x} - e^{-x} + 4e^{-2x}$$

For each nonhomogeneous equation, determine whether the method of undetermined coefficients could be used to find a particular solution.

(a) $y'' + 2xy' + x^2 y = \cos(2x)$

No, the left side is not constant coefficient.

(b) $y'' + 2y' + y = x^2 e^x$

Yes, the left is constant coef. and the right is from the allowable class of functions

For each nonhomogeneous equation, determine whether the method of undetermined coefficients could be used to find a particular solution.

(c) $y'' + 2y' + y = \dfrac{e^x}{x^2}$

No, the right side is not from the allowable class of functions

(d) $x^2 y'' - 4xy' + 6y = -x^2$

No, not constant coefficient.

A 200 gallon aquarium is full of water into which 10 lbs of salt is dissolved. Brine containing 1 lb/gal of salt is pumped in at a rate of 4 gallons per minute, and the well mixed solution is pumped out at the faster rate of 5 gallons per minute. Determine the number of pounds of salt $A(t)$ in the tank at time t in minutes, and indicated the interval over which the solution is valid.

$r_i = 4 \ \frac{\text{gal}}{\text{min}} \qquad c_i = 1 \ \frac{\text{lb}}{\text{gal}}$

$$\frac{dA}{dt} = r_i c_i - r_o c_o$$

$r_o = 5 \ \frac{\text{gal}}{\text{min}} \qquad c_o = \frac{A}{V}$

$$c_o = \frac{A}{V(0) + r_i t - r_o t} = \frac{A}{200 - t}$$

$A(0) = 10$

$$\frac{dA}{dt} = 4.1 - 5\frac{A}{200-t}, \quad A(0) = 10$$

$$\frac{dA}{dt} + \frac{5}{200-t}A = 4 \qquad P(t) = \frac{5}{200-t}$$

$$\mu = e^{\int \frac{5}{200-t}\,dt} = e^{-5\ln(200-t)} = e^{\ln(200-t)^{-5}} = (200-t)^{-5}$$

$$\frac{d}{dt}\left[(200-t)^{-5}A\right] = 4(200-t)^{-5}$$

$$\int \frac{d}{dt}\left[(200-t)^{-5} A\right] dt = \int 4\,(200-t)^{-5}\, dt$$

$$(200-t)^{-5} A = 4\,\frac{(200-t)^{-4}}{-4} + C$$

$$(200-t)^{-5} A = (200-t)^{-4} + C$$

$$A(t) = 200-t + C\,(200-t)^{5}$$

$$A(0) = 10 = 200 + C\,(200)^{5}$$

$$C(200)^5 = -190 \Rightarrow C = \frac{-190}{(200)^5}$$

$$\boxed{\text{So} \quad A(t) = 200 - t - \frac{190}{(200)^5}(200-t)^5 \quad \text{for } 0 \le t \le 200}$$

When $t \geq 200$, the tank is empty