July 21 Math 2254 sec 001 Summer 2015

Section 8.8: Power Series

Theorem: Let $\sum a_n(x-c)^n$ have positive radius of convergence R, and let the function f be defined by this power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Then f is differentiable on $(c-R, c+R)$. Moreover,

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}.$$

Theorem Continued

Moreover, f can be integrated term by term

$$\begin{aligned} \int f(x)\,dx &= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots \\ &= C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} \end{aligned}$$

The radius of convergence for each of these series is R.

Finding Power Series Representations

Find a power series representation for $f(x)$, and state the interval of convergence.

$$f(x) = \frac{1}{(1-x)^2}$$

Recall $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

for $-1 < x < 1$

Note $\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2}\cdot(-1) = \frac{1}{(1-x)^2}$

So for $-1 < x < 1$

$$f(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \ldots\right)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

Finding Power Series Representations

Find a power series representation for $g(x)$, and state the interval of convergence.

$$g(x) = \tan^{-1} x$$

Recall $\dfrac{1}{1+x^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$

for $-1 < x < 1$

For $-1 < x < 1$

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx$$

$$= \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\tan^{-1} x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$\tan^{-1} 0 = 0$

To find C, let $x = 0$

$$\tan^{-1} 0 = C + 0 - \frac{0^3}{3} + \dots \implies 0 = C$$

So $$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 < x < 1$$

Section 8.9: Taylor and Maclaurin Series

Suppose f has a power series representation for $|x-c|<R$. Try to determine a relationship between the coefficients a_n and the values of f and its derivatives as $x=c$.

$$f(x)=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+a_4(x-c)^4+a_5(x-c)^5+\cdots$$

$$f(c)=a_0+a_1(c-c)+a_2(c-c)^2+\cdots = a_0$$

$$\Rightarrow \boxed{a_0=f(c)}$$

$$f'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+4a_4(x-c)^3+5a_5(x-c)^4+\cdots$$

$$f'(c)=a_1+0+0\ldots \Rightarrow \boxed{a_1=f'(c)}$$

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + a_5(x-c)^5 + \cdots$$

$$f''(x) = 2a_2 + 3\cdot 2a_3(x-c) + 4\cdot 3a_4(x-c)^2 + 5\cdot 4a_5(x-c)^3 + \dots$$

$$f''(c) = 2a_2 + 0 + 0 + \dots \Rightarrow \boxed{a_2 = \frac{f''(c)}{2}}$$

$$f'''(x) = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(x-c) + 5\cdot 4\cdot 3a_5(x-c)^2 + \dots$$

$$f'''(c) = 3\cdot 2a_3 + 0 + 0 + \dots \Rightarrow \boxed{a_3 = \frac{f'''(c)}{2\cdot 3}}$$

$$a_0 = \frac{f(c)}{0!}, \quad a_1 = \frac{f'(c)}{1!}, \quad a_2 = \frac{f''(c)}{2!}.$$

$$a_3 = \frac{f'''(c)}{3!}.$$

In general $$a_n = \frac{f^{(n)}(c)}{n!}.$$

Theorem

Theorem: If f has a power series representation (a.k.a. *expansion*) centered at c,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n, \quad \text{for} \quad |x-c| < R,$$

then the coefficients are given by the formula

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Remark This notation makes use of the traditional convention that the *zeroth* derivative of f is f itself. That is,

$$\frac{f^{(0)}(c)}{0!} = f(c) = a_0.$$

The Taylor Series

Definition: If f has a power series representation centered at c, we can write it as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \end{aligned}$$

This is called the **Taylor series of f centered at c** (or **at c** or **about c**).

Definition: If $c = 0$, the series is called the **Maclaurin series of f**. In this case, the series above appears as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Example

Determine the Maclaurin series for $f(x) = e^x$. Find its radius of convergence.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(x) = e^x$, $f(0) = e^0 = 1$

$f'(x) = e^x$, $f'(0) = e^0 = 1$

$f''(x) = e^x$ $\vdots$

$f'''(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x$ $f^{(n)}(0) = e^0 = 1$

$\Rightarrow$ $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Use the ratio test to find the radius of convergence.

$$\lim_{n\to\infty} \left| \frac{x^{\cancel{n}+1}}{(n+1)!} \cdot \frac{n!}{\cancel{x^n}} \right| = \lim_{n\to\infty} \left| \frac{x \, \cancel{n!}}{\cancel{n!}(n+1)} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \qquad L = 0 < 1 \text{ for all real } x$$

$R = \infty$. The interval of convergence is

$$(-\infty, \infty).$$

e^x Approximated by terms in its Maclaurin Series

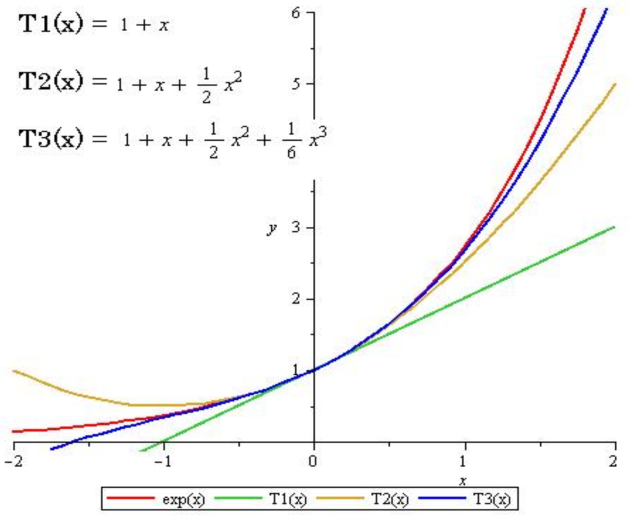

Figure: Plot of f along with the first 2, 3, and 4 terms of the Maclaurin series.

Taylor Polynomials

Definition: Suppose f is at least n times differentiable at $x = c$. The n^{th} **degree Taylor Polynomial of** f **centered at** c, denoted by T_n, is defined by

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n. \end{aligned}$$

Remark: Note that if f has a Taylor series centered at c, then the Taylor polynomials are what you get if you just take a finite number of terms, and discard the rest.

Remark: A Taylor **series** is like a *polynomial of infinite degree*, but a Taylor **polynomial** will have a well defined finite degree.

Example

Write out the first four Taylor polynomials of $f(x) = e^x$ centered at zero.

From before $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (Maclaurin Series)

$$T_0(x) = \frac{x^0}{0!} = 1 \qquad T_2(x) = 1 + x + \frac{x^2}{2!}$$

$$T_1(x) = 1 + \frac{x^1}{1!} = 1 + x \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Example

Find the Taylor polynomial of degree $n = 4$ centered at $c = 1$ for $g(x) = e^{3x}$.

$$T_4(x) = \sum_{k=0}^{4} \frac{g^{(k)}(1)}{k!}(x-1)^k$$

$$= \frac{g(1)}{0!} + \frac{g'(1)}{1!}(x-1) + \frac{g''(1)}{2!}(x-1)^2 + \frac{g'''(1)}{3!}(x-1)^3 + \frac{g^{(4)}(1)}{4!}(x-1)^4$$

$g(x) = e^{3x}$, $g(1) = e^3$

$g'(x) = 3e^{3x}$, $g'(1) = 3e^3$

$g''(x) = 3^2 e^{3x}$, $g''(1) = 9e^3$

$g'''(x) = 3^3 e^{3x}$, $g'''(1) = 27e^3$

$g^{(4)}(x) = 3^4 e^{3x}$, $g^{(4)}(1) = 81e^3$

$$T_4(x) = e^3 + 3e^3(x-1) + \frac{9e^3}{2}(x-1)^2 + \frac{27e^3}{6}(x-1)^3 + \frac{81e^3}{24}(x-1)^4$$

$$\boxed{T_4(x) = e^3 + 3e^3(x-1) + \frac{9e^3}{2}(x-1)^2 + \frac{9e^3}{2}(x-1)^3 + \frac{27e^3}{8}(x-1)^4}$$

Well Known Series and Results

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all real} \quad x$$

A consequence of this is:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

And with the radius of convergence being infinite, the following limit is useful:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{for every real number} \quad x$$

Maclaurin Series for $\sin x$

Derive the Maclaurin series of $f(x) = \sin x$. Find its radius of convergence.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(x) = \sin x$, $f(0) = \sin(0) = 0$

$f'(x) = \cos x$, $f'(0) = \cos(0) = 1$

$f''(x) = -\sin x$, $f''(0) = -\sin(0) = 0$

$f'''(x) = -\cos x$, $f'''(0) = -\cos(0) = -1$

$f^{(4)}(x) = \sin x$, $f^{(4)}(0) = 0 \ \ldots$

$$\sin x = \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

To get odd's use the formula $2n+1$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

To find the radius of convergence, use ratio test.

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}\, x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{\cancel{(-1)^{n}}\, x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\, x^{\cancel{2n+1}+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{\cancel{x^{2n+1}}} \right|$$

$$= \lim_{n\to\infty} \frac{x^2\, \cancel{(2n+1)!}}{\cancel{(2n+1)!}\,(2n+2)(2n+3)}$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0$$

$L = 0 < 1$ for all real x

The series converges for all real x.

The radius $R=\infty$, the

interval is $(-\infty, \infty)$.

Maclaurin Series for cos x

Use the fact that $\cos x = \frac{d}{dx} \sin x$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \frac{d}{dx} \sin x = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[(2n+1) x^{2n+1-1} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) x^{2n}$$

$$(2n+1)! = (2n)!\,(2n+1)$$

$$\Rightarrow \boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Well Known Series and Results

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all x} \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for all x} \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \text{for all x} \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad -1 < x \leq 1 \\
\tan^{-1} x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 \leq x \leq 1
\end{aligned}$$

Compositions, Products and Quotients

If we stay well within the radius of convergence, we can form compositions, products and quotients with Taylor and Maclaurin series.

Example: Find a Maclaurin series for $f(x) = e^{-x^2}$.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \quad \text{for all real } t$$

Set $t = -x^2$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$(-x^2)^n = (-1 \cdot x^2)^n$$

$$= (-1)^n (x^2)^n$$

$$= (-1)^n x^{2n}$$