July 23 Math 2254 sec 001 Summer 2015

Section 8.9: Taylor and Maclaurin Series

Definition: If f has a power series representation centered at c, we can write it as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \end{aligned}$$

This is called the **Taylor series of f centered at** c (or **at** c or **about** c).

Definition: If $c=0$, the series is called the **Maclaurin series of f**. In this case, the series above appears as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Taylor Polynomials

Definition: Suppose f is at least n times differentiable at $x = c$. The n^{th} **degree Taylor Polynomial of** f **centered at** c, denoted by T_n, is defined by

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n. \end{aligned}$$

Remark: Note that if f has a Taylor series centered at c, then the Taylor polynomials are what you get if you just take a finite number of terms, and discard the rest.

Remark: A Taylor **series** is like a *polynomial of infinite degree*, but a Taylor **polynomial** will have a well defined finite degree.

Well Known Series and Results

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all x} \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for all x} \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \text{for all x} \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad -1 < x \le 1 \\
\tan^{-1} x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 \le x \le 1
\end{aligned}
$$

Compositions, Products and Quotients

If we stay well within the radius of convergence, we can form compositions, products and quotients with Taylor and Maclaurin series.

Example: Find a Maclaurin series for $f(x) = \cos\sqrt{x}$ for $x > 0$.

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \quad \text{for all real } t$$

Set $t = \sqrt{x}$

$$\cos\sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!}$$

Note $\left(\sqrt{x}\right)^{2n} = \left[\left(\sqrt{x}\right)^2\right]^n = x^n$

$$\Rightarrow \quad \cos\sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} \qquad \text{for} \quad x > 0$$

Compositions, Products and Quotients

Example:

Find a Maclaurin series representation for the indefinite integral.

$$\int \sin x^2 \, dx$$

we're finding an indefinite integral, so there will remain a "+ C."

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \quad \text{for all real } t$$

Set $t = x^2$, $\quad \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$

$$(x^2)^{2n+1} = x^{2(2n+1)} = x^{4n+2}$$

$$\int \sin x^2 \, dx = \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \right) dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)!} \int x^{4n+2} \, dx \right]$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{x^{4n+2+1}}{4n+2+1} \right)$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$$

Compositions, Products and Quotients

Example:

Use the Maclaurin series for e^x to find a Taylor series for $f(x) = e^x$ centered at $c = -1$.

Note $e^x = e^{x+1-1} = e^{(x+1)} \cdot e^{-1}$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \quad \text{for all real } t$$

Set $t = x+1$

$$f(x) = e^x = e^{-1} e^{(x+1)} = e^{-1} \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{-1}(x+1)^n}{n!}$$

Theorem: The Binomial Series

Theorem: For k any real number and $|x| < 1$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Here $\binom{k}{n}$ is read as k *choose* n. If is defined by

$k-(n-1)$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

If k is a positive integer, this has the traditional meaning

$$\binom{k}{n} = \frac{k!}{(k-n)!n!}.$$

Example

Use the Binomial series to find the Taylor polynomial of degree 3 centered at zero for

$$f(x) = \frac{1}{\sqrt[3]{1+x}} = (1+x)^{-1/3} \qquad k = \frac{-1}{3} \text{ here}$$

$$T_3(x) = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3$$

$$\frac{k(k-1)}{2!} = \frac{\frac{-1}{3}\left(\frac{-1}{3}-1\right)}{2} = \frac{\frac{-1}{3}\left(\frac{-4}{3}\right)}{2} = \frac{\frac{4}{9}}{2} = \frac{2}{9}$$

$$\frac{k(k-1)(k-2)}{3!} = \frac{\frac{-1}{3}\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{6} = \frac{\frac{-1}{3}\left(\frac{-4}{3}\right)\left(\frac{-7}{3}\right)}{6} = \frac{\frac{-28}{27}}{6} = \frac{-14}{81}$$

$$\boxed{T_3(x) = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3}$$

One More Example

Suppose we have the Taylor series for a function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{2^n (n+2)}$.

Use this to evaluate the following derivative of f:

$$f^{(5)}(3)$$

Recall $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$

$a_5 = \frac{f^{(5)}(3)}{5!} \Rightarrow f^{(5)}(3) = 5!\, a_5$

here, $a_5 = \frac{(-1)^5}{2^5(5+2)} = \frac{-1}{32 \cdot 7}$

$f^{(5)}(3) = 5! \left(\frac{-1}{32 \cdot 7}\right) = \frac{-1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{32 \cdot 7} = \frac{-15}{28}$

Section 8.10: Applications of Taylor Expansions

Recall: The Taylor polynomial of degree n centered at c shares the value of f at c and has the same first n derivative values as f does at the center. Hence T_n approximates the function f—typically, the higher the value of n, and closer we stay to the center, the better the approximation is.

We can exploit the *nice* nature of polynomials if f itself is somehow difficult to manage!

▸ Wikipedia Page w/ Some Nice Graphics

Example

Approximate the value of $\sqrt[3]{9}$ by using an appropriate Taylor polynomial of degree 2.

For $x \approx c$, $\quad f(x) \approx T_2(x)$

We want to approximate a cube root, so take $f(x) = \sqrt[3]{x}$. The closest perfect cube to 9 is 8. Take $c = 8$.

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$f(x) = x^{1/3}$, $f(8) = 8^{1/3} = 2$

$f'(x) = \frac{1}{3} x^{-2/3}$ $f'(8) = \frac{1}{3} \frac{1}{8^{2/3}} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

$f''(x) = \frac{1}{3} \cdot \frac{-2}{3} x^{-5/3}$ $f''(8) = \frac{-2}{9} \frac{1}{8^{5/3}} = \frac{-2}{9} \cdot \frac{1}{32} = \frac{-1}{144}$

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{\frac{1}{144}}{2}(x-8)^2$$

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

$$f(9) = \sqrt[3]{9} \approx T_2(9)$$

$$T_2(9) = 2 + \frac{1}{12}(9-8) - \frac{1}{288}(9-8)^2$$

$$= 2 + \frac{1}{12} - \frac{1}{288}$$

$$= \frac{576 + 24 - 1}{288} = \frac{599}{288}$$

$$\sqrt[3]{9} \approx \frac{599}{288}$$

Graph of f and T_2 Approximation

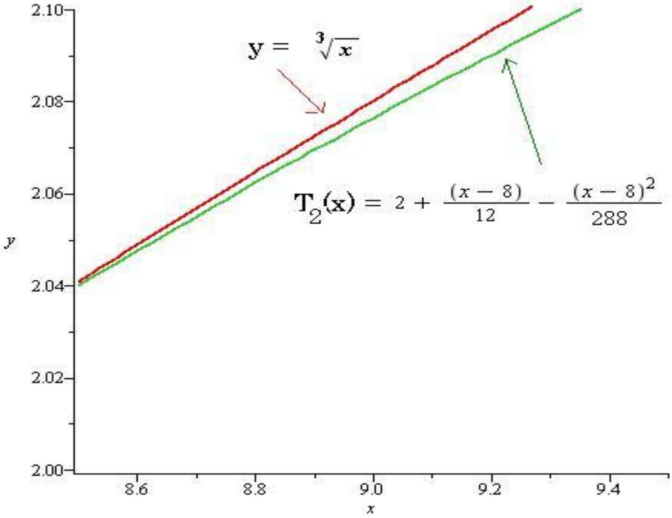

Figure: $f(x) = \sqrt[3]{x}$ together with the second degree Taylor polynomial near the point being approximated.

Example

Find the Taylor polynomial of degree 2 centered at $\frac{\pi}{2}$ for $f(x) = \sin x$.
Use this to find an approximation to $\sin 80^\circ$

$$T_2(x) = f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x-\frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2$$

$f(x) = \sin x \qquad f(\pi/2) = 1$

$f'(x) = \cos x \qquad f'(\pi/2) = 0$

$f''(x) = -\sin x \qquad f''(\pi/2) = -1$

$$T_2(x) = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

$$\sin 80^\circ = \sin\left(80 \cdot \frac{\pi}{180}\right) = \sin\left(\frac{4\pi}{9}\right)$$

$$\approx T_2\left(\frac{4\pi}{9}\right)$$

$$= 1 - \frac{1}{2}\left(\frac{4\pi}{9} - \frac{\pi}{2}\right)^2$$

$$= 1 - \frac{1}{2}\left(\frac{8\pi - 9\pi}{18}\right)^2 = 1 - \frac{1}{2}\,\frac{(-\pi)^2}{18^2}$$

$$= 1 - \frac{\pi^2}{648} = \frac{648 - \pi^2}{648}$$

Graph of f and Polynomial Approximations

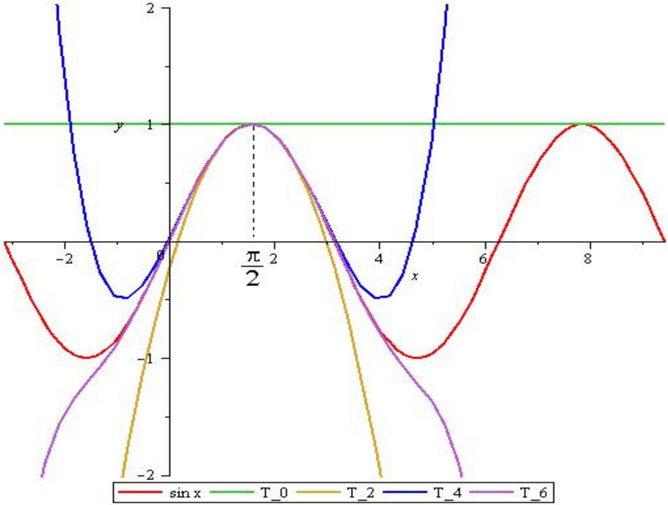

Figure: $f(x) = \sin x$ together with T_n $(n = 0, 2, 4, 6)$ centered at $\frac{\pi}{2}$.

Approximating a Definite Integral

Use the first two nonzero terms of the Maclaurin series for $\frac{1}{1+x^3}$ to approximate the integral

$$\int_0^{0.1} \frac{dx}{1+x^3}$$

$a=1 \quad r=-x^3$

Get the power series

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n} \quad \text{for } |-x^3| < 1$$

$$= 1 - x^3 + x^6 - x^9 + \dots$$

$$\int_0^{0.1} \frac{dx}{1+x^3} \approx \int_0^{0.1} (1-x^3)\, dx$$

$$= x - \frac{x^4}{4} \Big|_0^{0.1}$$

$$= 0.1 - \frac{(0.1)^4}{4} - 0$$

$$= 0.1 - 0.25(0.0001)$$

$$= 0.1 - 0.000025$$

$$= 0.099975$$

$$1 + x^3 = (1+x)(1 - x + x^2)$$

Using Taylor Series to Compute Limits

Use the Maclaurin series for $\sin x$ to verify the well known limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

For $x \approx 0$, $x \neq 0$

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right)$$

$$= 1$$

Example

Use an appropriate Taylor series to evaluate the limit

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\cos x - 1 + \frac{x^2}{2} = \left(\cancel{1} - \cancel{\frac{x^2}{2!}} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots\right) - \cancel{1} + \cancel{\frac{x^2}{2}}$$

$$\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{\frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots}{x^4}$$

$$= \frac{1}{4!} - \frac{x^2}{6!} + \frac{x^4}{8!} - \cdots$$

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \lim_{x \to 0} \left(\frac{1}{4!} - \frac{x^2}{6!} + \frac{x^4}{8!} - \cdots \right)$$

$$= \frac{1}{4!} = \frac{1}{24}$$