July 2 Math 2254 sec 001 Summer 2015

Section 8.1: Sequences

Definition: A **sequence** is a function whose domain is a subset of the integers and whose range is a subset of the real numbers.

Some examples we've seen include:

- $\left\{\frac{2n}{n+1}\right\}_{n=1}^{\infty}$ with terms $1, \frac{4}{3}, \frac{3}{2}, \ldots$
- $\{(-1)^n\}_{n=0}^{\infty}$ with terms $1, -1, 1, \ldots$, and
- $f_0 = 1, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 2,$ with terms $1, 1, 2, 3, 5, \ldots$

Limits and Convergence

Definition: A sequence $\{a_n\}$ is said to be **convergent** with limit L provided

$$\lim_{n\to\infty} a_n = L.$$

A sequence that is not convergent is **divergent.**

Example: Determine if the sequence $a_n = \frac{2n}{n+1}$ is convergent or divergent. If convergent, determine the limit.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{n+1}$$

$$= \lim_{n\to\infty} \left(\frac{2n}{n+1}\right) \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

* Recall

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

for all powers $p > 0$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{1}{n}}$$

$$= \frac{2}{1+0} = 2$$

The sequence is convergent with limit $L = 2$.

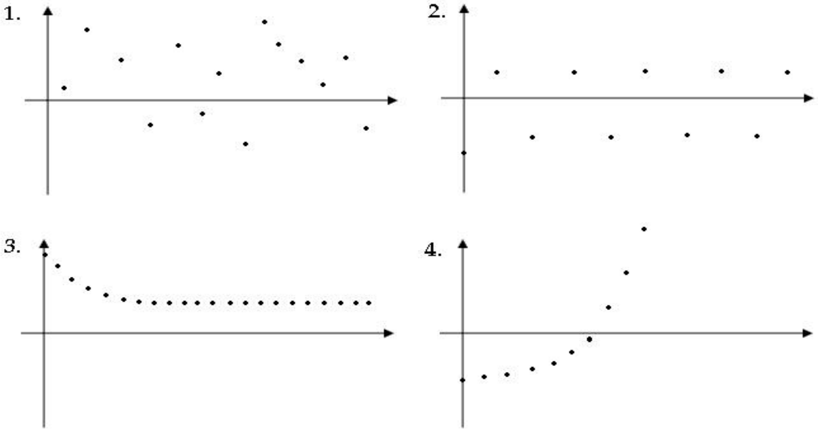

Figure: Plotted as points (n, a_n), sequence terms may jump around (1), oscillate back and forth between two or more values (2), converge to a limit (3), or become unbounded going to $+\infty$ or $-\infty$ (4).

Examples

Determine if the sequence is convergent or divergent. If convergent, find its limit.

(a) $\{(-1)^n\}_{n=0}^{\infty}$ Let $a_n = (-1)^n$

$a_0 = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, ...

$\lim_{n\to\infty} (-1)^n$ DNE

The sequence is divergent.

(it oscillates)

(b) $\{2^n\}_{n=0}^{\infty}$ Let $a_n = 2^n$

$a_0 = 2^0 = 1, \quad a_1 = 2^1 = 2, \quad a_2 = 2^2 = 4, \ldots$

$$\lim_{n\to\infty} 2^n = \infty$$

The sequence is divergent.

(c) $\left\{\frac{1}{\ln n}\right\}_{n=2}^{\infty}$ Let $a_n = \frac{1}{\ln n}$

$$a_2 = \frac{1}{\ln 2}, \quad a_3 = \frac{1}{\ln 3}, \ldots$$

$$\lim_{n\to\infty} \frac{1}{\ln n} = 0$$

The sequence is convergent with limit 0.

Limit Laws for Sequences

Theorem: Suppose $\{a_n\}$ and $\{b_n\}$ are convergent to A and B, respectively, and let c be constant. Then

$$
\begin{aligned}
\lim_{n\to\infty}(a_n \pm b_n) &= A \pm B \\
\lim_{n\to\infty} ca_n &= cA \\
\lim_{n\to\infty}(a_n b_n) &= AB \\
\lim_{n\to\infty}\frac{a_n}{b_n} &= \frac{A}{B} \quad \text{if} \quad b_n \neq 0, \quad B \neq 0 \\
\lim_{n\to\infty}[a_n]^p &= A^p \quad \text{if} \quad p > 0, \quad a_n \geq 0
\end{aligned}
$$

Example

Use appropriate limit laws to determine the limit if it exists.

(a) $\left\{\sqrt{1-\dfrac{1}{2^n}}\right\}$

Recall $\lim_{n\to\infty} 2^n = \infty$

so $\lim_{n\to\infty} \dfrac{1}{2^n} = 0$.

Hence $\lim_{n\to\infty}\left(1-\frac{1}{2^n}\right) = 1-0 = 1$

Finally $\lim_{n\to\infty}\sqrt{1-\frac{1}{2^n}} = \sqrt{1-0} = \sqrt{1} = 1$

Theorem (on continuous functions)

Theorem: If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Example: Determine the limit

$$\lim_{n\to\infty} \exp\left(\frac{1}{n^2}\right)$$

$= \exp(0)$

$= e^0 = 1$

Note $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Also $f(x) = e^x$

is continuous @ 0.

Using Functions of a Real Variable

Definition: A function f will be called a **related function** for the sequence $\{s_n\}$ provided its domain is $(0,\infty)$ (or $[0,\infty)$) and if

$$f(n) = s_n \quad \text{for each } n \text{ in the domain of } s_n.$$

Examples: $\{s_n\} = \{e^{-n}\}$ has related function $f(x) = e^{-x}$.

$\{a_n\} = \left\{\frac{n+1}{2^n}\right\}$ has related function $f(x) = \frac{x+1}{2^x}$.

A Word of Caution about Derivatives

Remember that if $f'(x)$ exists, it is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

If we have a function $\{s_n\}$ whose domain is $1, 2, 3, \ldots$, then $n + h$ is not in the domain of $\{s_n\}$ if h is not a positive integer.

That is, s_{n+h} DOES NOT MAKE SENSE IF h IS NOT AN INTEGER!!

We can't take a derivative of a sequence. But, we might be able to take a derivative of a related function.

Theorem

Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for each integer n, then $\lim_{n\to\infty} a_n = L$.

Example: Determine the limit of the sequence $\left\{\frac{\ln n}{n}\right\}$.

$\lim_{n\to\infty} \frac{\ln(n)}{n} = $ "$\frac{\infty}{\infty}$" indeterminate form

Let $f(x) = \frac{\ln x}{x}$ so f is the related function

for a_n

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"}$$

Use l'Hospital's rule

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0$$

Hence $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

Squeeze Theorem

Theorem: Suppose $a_n \leq b_n \leq c_n$ for all $n \geq n_0$. If

$$\lim_{n\to\infty} a_n = L \quad \text{and} \quad \lim_{n\to\infty} c_n = L, \quad \text{then} \quad \lim_{n\to\infty} b_n = L.$$

Corollary: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Note $\quad -|a_n| \leq a_n \leq |a_n|$

The Squeeze Theorem

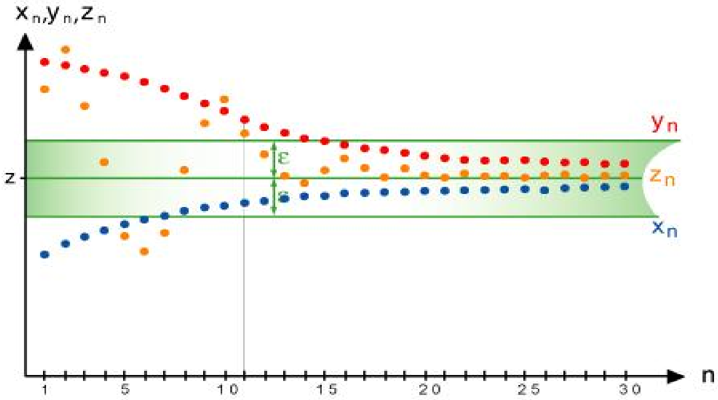

Figure: The sequence $\{z_n\}$ (orange) is *squeezed* between the sequences $\{x_n\}$ (blue) and $\{y_n\}$ (red) for all $n \geq 11$. Since $x_n \to z$ and $y_n \to z$, it is guaranteed that $z_n \to z$.

Factorials

For an integer $n \geq 1$ the expression $n!$, read *n factorial* is defined as the product of the first n integers. That is

$$n! = 1 \cdot 2 \cdot 3 \cdots n. \quad \text{Also} \quad 0! = 1.$$

Examples: Compute 4! and 7!.

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

$$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$$

Show that $(n+1)! = n!(n+1)$.

$$(n+1)! = \underbrace{1 \cdot 2 \cdot 3 \cdots n}_{n!} \cdot (n+1) = n!\,(n+1)$$

Squeeze Theorem Example

Show that $0 \leq a_n \leq \frac{1}{n}$ and comment on the convergence or divergence of the sequence

$$a_n = \frac{n!}{n^n}.$$

$$a_n = \frac{n!}{n^n} = \frac{\overbrace{1\cdot 2\cdot 3\cdot 4\cdots n}^{n \text{ factors}}}{\underbrace{n\cdot n\cdot n\cdot n\cdots n}_{n \text{ factors}}}$$

For example

$$a_1 = \frac{1!}{1^1} = \frac{1}{1} = 1$$

$$a_2 = \frac{2!}{2^2} = \frac{1\cdot 2}{4} = \frac{1}{2}$$

$$a_3 = \frac{3!}{3^3} = \frac{1\cdot 2\cdot 3}{27} = \frac{2}{9}, \ldots$$

$$a_n = \frac{1\cdot 2\cdot 3\cdot 4\cdots n}{n\cdot n\cdot n\cdot n\cdots n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\left(\frac{4}{n}\right)\cdots\left(\frac{n}{n}\right)$$

$$\leq \frac{1}{n}\cdot 1\cdot 1\cdot 1\cdots 1 \quad = \frac{1}{n}$$

That is, $\frac{n!}{n^n} \leq \frac{1}{n}$ for all $n \geq 1$

Since $n! > 0$ and $n^n > 0$ for all $n \geq 1$

$$\frac{n!}{n^n} > 0$$

Together, we have $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$

for all $n \geq 1$.

$\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$

Hence $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ by the squeeze theorem.

A Special Sequence

Let r be a real number. Determine the convergence or divergence of the sequence

$$a_n = r^n.$$

Cases: $r=1$, $r=-1$, $-1<r<1$, and $r>1$ or $r<-1$

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Case 1: $r=1$, $a_n = 1^n = 1$ for all n

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 = 1$$

Convergent w/ limit $L=1$

Case 2: $r = -1$, $a_n = (-1)^n = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}$

The sequence is divergent.

Case 4: $r > 1$ or $r < -1$

$\lim_{n\to\infty} r^n$ if $r > 1$ $\lim_{n\to\infty} r^n = \infty$

if $r < -1$ $\lim_{n\to\infty} r^n$ DNE

The sequence is divergent.

Case 3: $-1 < r < 1$ i.e. $|r| < 1$

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

Hence $\lim_{n\to\infty} r^n = 0$

The sequence converges w/ $L = 0$.

Monotone Sequences

Definition: A sequence $\{a_n\}$ is said to be **nondecreasing** if $a_{n+1} \geq a_n$ for each n. It is said to be **increasing** if $a_{n+1} > a_n$ for each n.

For example, $\{2^n\}$ is **increasing** because

$$a_{n+1} = 2^{n+1} = 2 \cdot 2^n > 2^n = a_n \quad \text{for every nonnegative integer } n.$$

Monotone Sequences

Definition: A sequence $\{a_n\}$ is said to be **nonincreasing** if $a_{n+1} \leq a_n$ for each n. It is said to be **decreasing** if $a_{n+1} < a_n$ for each n.

For example, $\left\{\frac{1}{n}\right\}$ is **decreasing** because

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n \quad \text{for every } n \geq 1.$$

Monotone Sequences

Definition: If a sequence is nondecreasing, increasing, nonincreasing, or decreasing, it is called **monotonic.**

So $\{2^n\}$, and $\left\{\frac{1}{n}\right\}$ are examples of monotonic sequences.

Is $\{1\}$ a monotonic sequence?

yes, it is both nondecreasing and nonincreasing

Is $\{(-1)^n\}$ a monotonic sequence?

no

Example Using algebra to determine if a sequence is monotone:

$\left\{ \dfrac{n}{n^2+1} \right\}$. Show that this is a decreasing sequence.

Note $a_n = \dfrac{n}{n^2+1}$, $a_{n+1} = \dfrac{n+1}{(n+1)^2+1}$

Note

$$(n+1)(n^2+1) = n^3+n^2+n+1 < n^3+2n^2+n+1$$

$$\leq n^3+2n^2+2n$$

$$= n(n^2+2n+2)$$

$$= n\left(n^2+2n+1+1\right)$$

$$= n\left((n+1)^2+1\right)$$

This gives

$$(n+1)(n^2+1) < n\left((n+1)^2+1\right)$$

divide by n^2+1 and $(n+1)^2+1$

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \quad \text{i.e.} \quad a_{n+1} < a_n$$

Hence the sequence is decreasing.

Example Using a related function to determine if a sequence is monotone:

$\left\{ \dfrac{n}{n^2+1} \right\}$. Show that this is a decreasing sequence.

Let $f(x) = \dfrac{x}{x^2+1}$ so f is the related function.

We'll show that f is decreasing.

$$f'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$f'(x) < 0$ for $x > 1$ since $1 - x^2 < 0$

Hence f is a decreasing function.

$$f(n+1) = \frac{n+1}{(n+1)^2+1} < f(n) = \frac{n}{n^2+1}$$

this point is to the right of this one

Hence $\left\{ \frac{n}{n^2+1} \right\}$ is a decreasing sequence.