July 3 Math 1190 sec. 51 Summer 2017

Section 5.1: Area (under the graph of a nonnegative function)

We're solving the problem of finding the area enclosed between the graph of a function f and the x-axis on the interval $[a,b]$ under the assumptions that

- f is continuous on the interval $[a,b]$, and
- f is nonnegative, i.e $f(x) \geq 0,$ on $[a,b]$.

Area as the Limit of Riemann Sums

- We made a partition $a = x_0 < x_1 < \cdots < x_n = b$,
- approximated the area of each piece with a rectangle of height $f(c_i)$ and width Δx
- approximate the whole area with the sum of the areas of the rectangles

$$A \approx \sum_{i=1}^{n} f(c_i)\Delta x$$

- then the true area is given by the limit

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

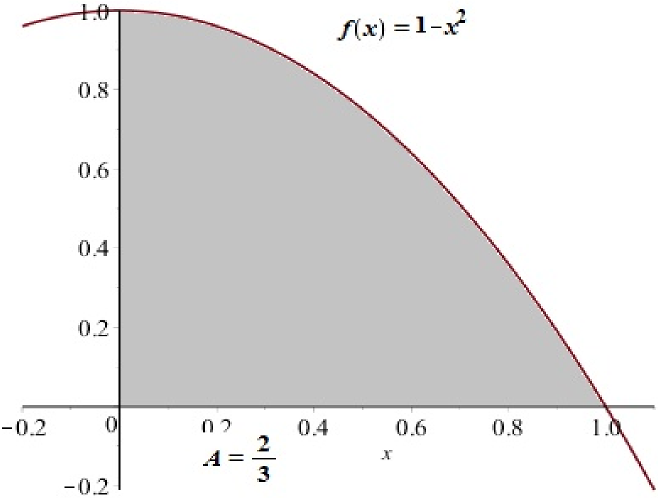
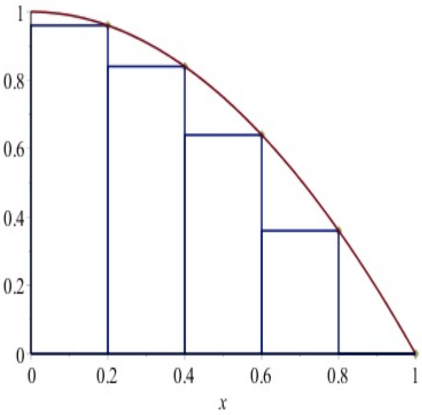


Figure: We found the area under the curve $f(x) = 1 - x^2$ over the interval $[0, 1]$. The area was $\frac{2}{3}$.



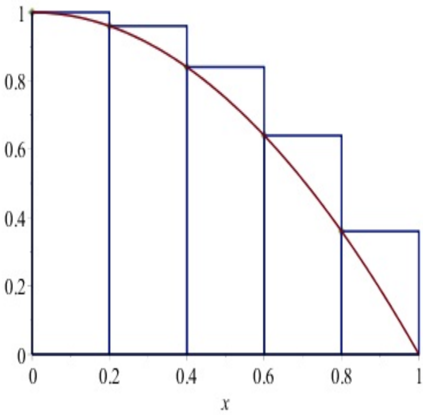


Figure: We can use right or left end points to define the rectangle heights.

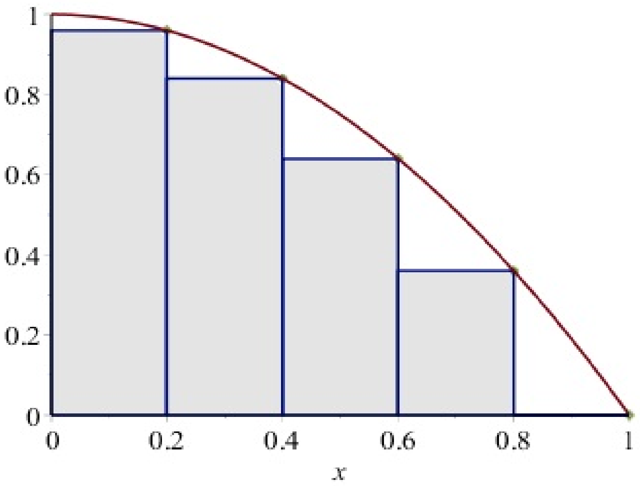


Figure: Here's the region with 5 rectangles using right end points. $A \approx \frac{14}{25}$

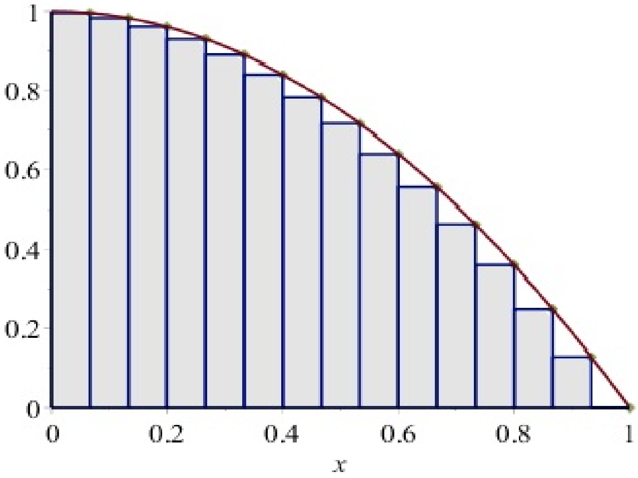


Figure: Here's the region with 15 rectangles using right end points. $A \approx \frac{427}{675}$ (for reference, the true area is $450/675$)

Riemann Sum Demo

▸ GeoGebra Riemann Sum Demo

Equally Spaced Partition Case:

- $\Delta x = \frac{b-a}{n}$
- $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, i.e. $x_i = a + i\Delta x$
- Taking heights to be

$$\text{left ends} \quad c_i = x_{i-1} \quad \text{area} \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$\text{right ends} \quad c_i = x_i \quad \text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

- The true area exists (for f continuous) and is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Section 5.2: The Definite Integral

We saw that a sum of the form

$$f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$$

approximated the area of a region if f was continuous and positive. And that under these conditions, the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty}[f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x]$$

was the value of this area.

Can we generalize this dropping the requirement that f is positive? that f is continuous?

Definition (Definite Integral)

Let f be defined on an interval $[a, b]$. Let

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b$$

be any partition of $[a, b]$, and $\{c_1, c_2, \ldots, c_n\}$ be any set of sample points. Then the **definite integral of** f **from** a **to** b is denoted and defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

provided this limit exists. Here, the limit is taken over all possible partitions of $[a, b]$.

Terms and Notation

- **Riemann Sum:** any sum of the form $f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$
- **Integral Symbol/Sign:** $\int$ (a stretched "S" for "sum")
- **Integrable:** If the limit does exists, f is said to be integrable on $[a, b]$
- **Limits of Integration:** a is called the lower limit of integration, and b is the upper limit of integration
- **Integrand:** the expression "$f(x)$" is called the integrand

- **Differential:** *dx* is called a differential, it indicates what the variable is and can be thought of as the limit of Δx (just as it is in the derivative notation "$\frac{dy}{dx}$").

- **Dummy Variable/Variable of Integration:** the variable that appears in both the integrand and in the differential. For example, if the differential is *dx*, the dummy variable is *x*; it the differential is *du*, the dummy variable is *u*

upper limit of integration

$$\int_a^b f(x)\,dx$$

lower limit

Important Remarks

(1) If the integral does exist, it is a **number**. That is, it does not depend on the dummy variable of integration. The following are equivalent.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(q)\,dq$$

(2) The definite integral is a limit of Riemann Sums!

(3) If f is positive and continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{ the area under the curve.}$$

Questions

Consider the integral $\int_{-3}^{\pi} f(r)\,dr$

(1) The dummy variable of integration is

(a) x

(b) r

(c) can't be determined without more information

(d) dr

Questions

(2) If it is known that $\int_{-3}^{\pi} f(r)\,dr = 7$, then

$$\int_{-3}^{\pi} f(x)\,dx =$$

(a) $7x$

(b) -7

(c) can't be determined without more information

(d) 7

What if f is continuous, but not always positive?

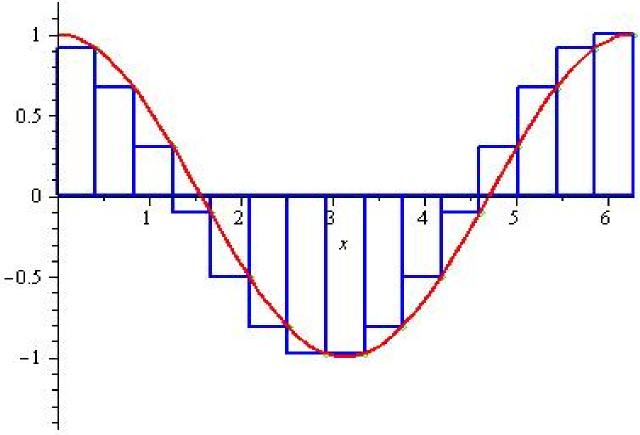


Figure: A function that changes signs on $[a, b]$. (Here, $f(x) = \cos x$, $a = 0$ and $b = 2\pi$; the partition has 15 subintervals.)

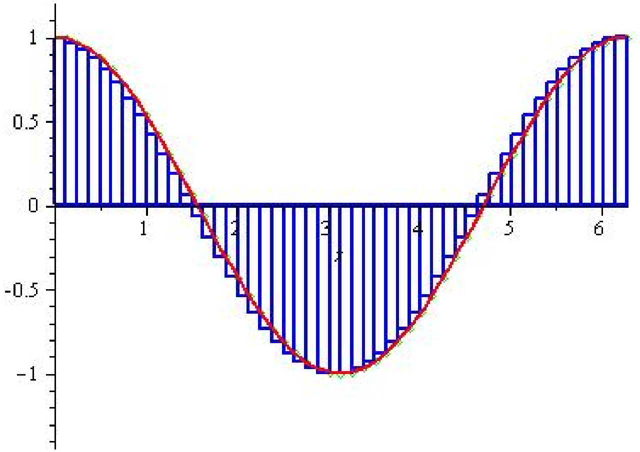


Figure: The same function but with 50 subintervals.

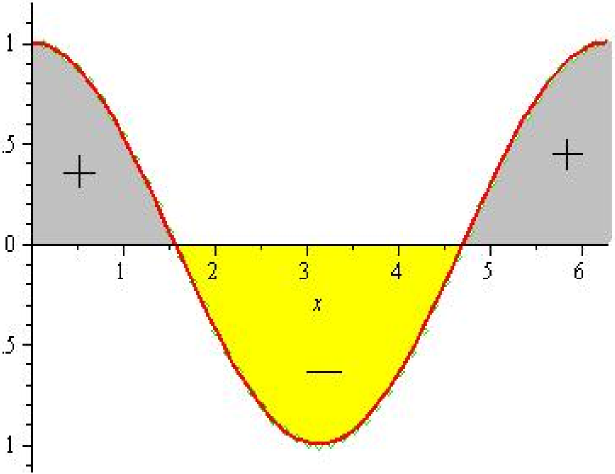


Figure: $\int_a^b f(x)\,dx =$ area of gray region $-$ area of yellow region

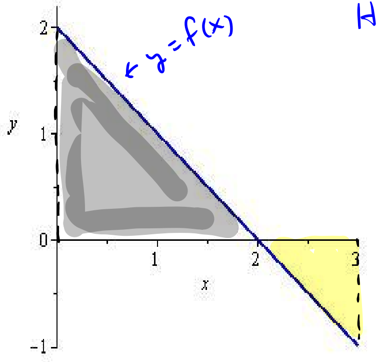
Another Important Remark

(4) If f is piecewise continuous enclosing region(s) of total area A_1 **above** the x-axis and enclosing region(s) of total area A_2 **below** the x-axis, then

$$\int_a^b f(x)\,dx = A_1 - A_2$$

Example

Use area to evaluate the integral $\int_0^3 (2 - x)\, dx$.



Here $f(x) = 2 - x$, the integrand.

$$\int_0^3 (2-x)\,dx = \text{Gray area} - \text{yellow area}$$

gray triangle $b = 2$, $h = 2$ Area $= \frac{1}{2}bh = \frac{1}{2}(2)(2)$

$= 2$

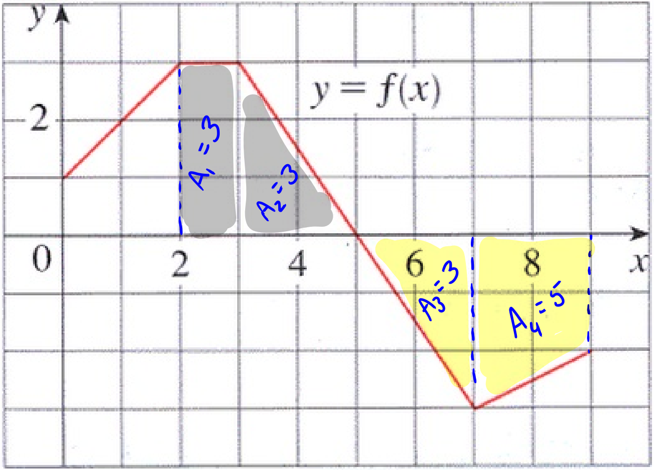
yellow triangle $b = 1$, $h = 1$

$$\text{Area} = \frac{1}{2} bh = \frac{1}{2}(1)(1) = \frac{1}{2}$$

$$\int_0^3 (2-x)\,dx = 2 - \frac{1}{2} = \frac{3}{2}$$

Example

Consider the graph of $y = f(x)$ shown.



Example

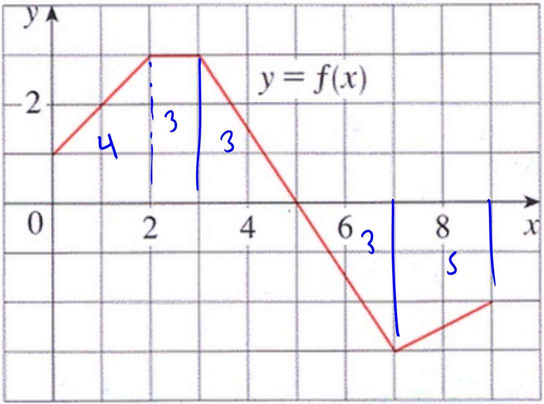
Use the graph on the preceding page to evaluate each integral.

$$\int_2^7 f(x)\,dx = A_1 + A_2 - A_3 = 3 + 3 - 3 = 3$$

$$\int_7^9 f(x)\,dx = -A_4 = -5$$

Question

Use the graph to evaluate $\int_0^9 f(x)\,dx$



(a) 6

(b) 2

(c) -2

(d) 4

Important Theorems:

Theorem: If f is continuous on $[a, b]$ or has only finitely many jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$

Theorem: If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

where

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad c_i = a + i\Delta x.$$

A couple of definitions:

Definition: If $f(a)$ is defined, then

$$\int_a^a f(x)\,dx = 0.$$

equal upper and lower limits

In particular, the integral of a continuous function over a single point is zero.

Definition: If $\int_a^b f(x)\,dx$ exists, then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Reversing the limits of integration negates the value of the integral.

Example

It can be shown that $\int_0^{\pi} \sin^2(x)\,dx = \frac{\pi}{2}$.

Evaluate

$$\int_{\pi}^{0} \sin^2(t)\,dt$$

$$= -\int_0^{\pi} \sin^2(t)\,dt$$

$$= -\frac{\pi}{2}$$

we have two differences

① the dummy x is replaced by t. ← no effect

② the limits are swapped

↑ negates the value

Question

Suppose it is known that $\int_3^{10} f(x)\,dx = -12$

Evaluate $\int_{10}^{3} f(x)\,dx$ $= -\int_3^{10} f(x)\,dx = -(-12) = 12$

(a) 12

(b) -12

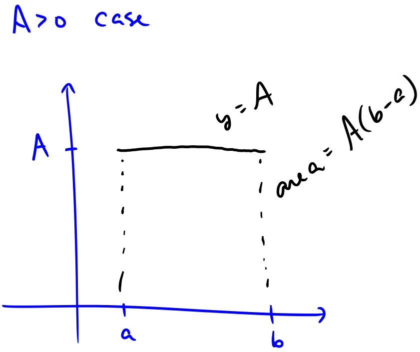
(c) $f(10)$

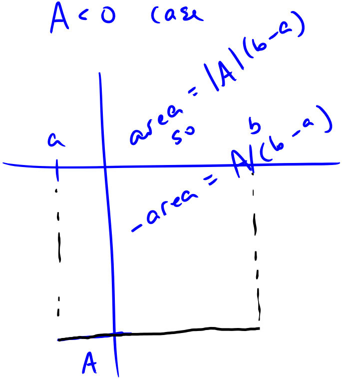
(d) can't be determined without more information

A simple integral

If $f(x) = A$ where A is any constant, then

$$\int_a^b f(x)\,dx = \int_a^b A\,dx = A(b-a).$$





Question

$$\int_3^7 \pi\, dx = \pi(7-3) = 4\pi$$

(a) 4π

(b) 7π

(c) 3π

(d) can't be determined without more information

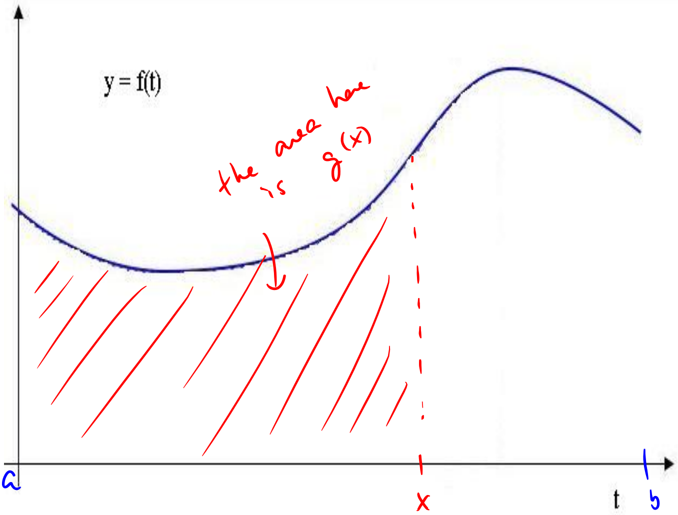
Section 5.3: The Fundamental Theorem of Calculus

Suppose f is continuous on the interval $[a, b]$. For $a \leq x \leq b$ define a new function

$$g(x) = \int_a^x f(t)\, dt$$

How can we understand this function, and what can be said about it?

Geometric interpretation of $g(x) = \int_a^x f(t)\,dt$



Figure

Theorem: The Fundamental Theorem of Calculus (part 1)

If f is continuous on $[a,b]$ and the function g is defined by

$$g(x)=\int_a^x f(t)\,dt \quad \text{for} \quad a\le x\le b,$$

then g is continuous on $[a,b]$ and differentiable on (a,b). Moreover

$$g'(x)=f(x).$$

This means that the new function g is an **antiderivative** of f on (a,b)!
"FTC" = "fundamental theorem of calculus"

Example:

Evaluate each derivative.

(a) $\displaystyle \frac{d}{dx}\int_0^x \sin^2(t)\,dt$ $= \sin^2(x)$

here, $f(t) = \sin^2(t)$

$a = 0$

(b) $\displaystyle \frac{d}{dx}\int_4^x \frac{t-\cos t}{t^4+1}\,dt$ $= \dfrac{x-\cos x}{x^4+1}$

here, $f(t) = \dfrac{t-\cos t}{t^4+1}$

$a = 4$

Question

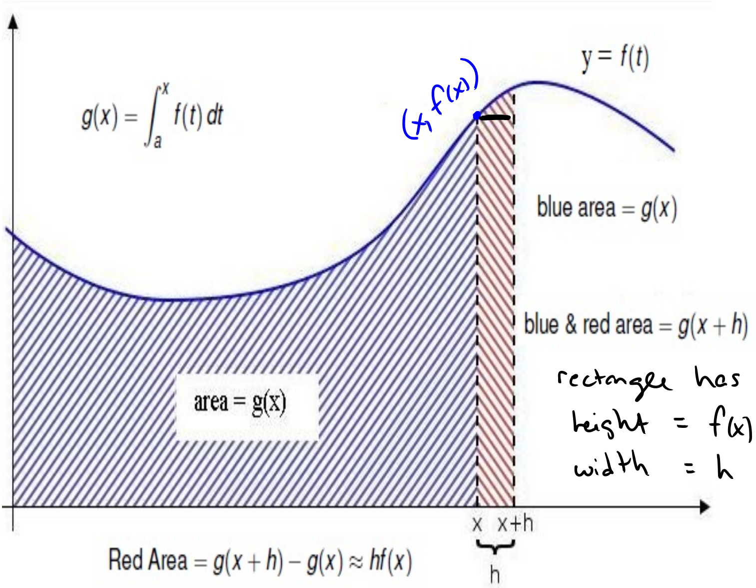
Evaluate $\frac{d}{dx}\int_2^x e^{3t^2}\,dt$

(a) e^{3x^2}

(b) $6xe^{3x^2}$

(c) $e^{3x^2} - e^{12}$

Geometric Argument of FTC



$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \approx \lim_{h \to 0} \frac{h f(x)}{h}$$

with the approximation getting better as h gets smaller.

So taking $h \to 0$

$$g'(x) = \lim_{h \to 0} f(x) = f(x)$$

Chain Rule with FTC

Evaluate each derivative.

(a) $\frac{d}{dx}\int_0^{x^2} t^3\,dt$

This is a composition with outside function

$F(u) = \int_0^u t^3\,dt$

and inside $u = x^2$

$F'(u) = u^3$ (FTC)

$u' = 2x$ (power rule)

$= (x^2)^3 \cdot (2x)$

$= x^6 \cdot (2x)$

$= 2x^7$

(b) $\frac{d}{dx}\int_x^7 \cos(t^2)\,dt$

For the FTC, we need the x as the upper limit.

We'll use

$$\int_x^7 \cos(t^2)\,dt = -\int_7^x \cos(t^2)\,dt$$

$$= \frac{d}{dx}\left(-\int_7^x \cos(t^2)\,dt\right)$$

$$= -\frac{d}{dx}\int_7^x \cos(t^2)\,dt$$

$$= -\cos(x^2)$$

Question

Use the chain rule where $f(u) = \int_1^u \sin^{-1} t \, dt$ and $u = 7x$ to evaluate

$$\frac{d}{dx}\int_1^{7x} \sin^{-1} t \, dt$$

$f'(u) = \sin^{-1} u$ by the FTC

$u' = 7$

so $f'(u)u' = \sin^{-1}(7x) \cdot 7$

(a) $\frac{1}{\sqrt{1-7x^2}}$

(b) $\sin^{-1}(7x)$

(c) $\frac{7}{\sqrt{1-49x^2}}$

(d) $7\sin^{-1}(7x)$

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

Since F can be any antiderivative, we usually take the simplest one – the one without "+C"

Example: Use the FTC to show that $\int_0^b x\,dx = \frac{b^2}{2}$

we need an antiderivative. Here $f(x) = x$.

Using the power rule we can take

$$F(x) = \frac{x^{1+1}}{1+1} = \frac{x^2}{2}$$

The FTC says

$$\int_0^b x\,dx = F(b) - F(0) = \frac{b^2}{2} - \frac{0^2}{2} = \frac{b^2}{2}$$

Notation

Suppose F is an antiderivative of f. We write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

or sometimes

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For example

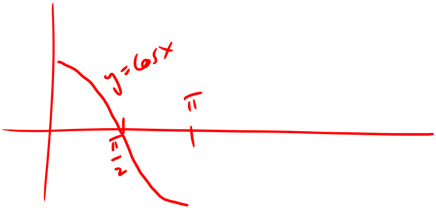
$$\int_0^b x\ dx = \frac{x^2}{2}\Big|_0^b = \frac{b^2}{2} - \frac{0^2}{2} = \frac{b^2}{2}$$

Evaluate each definite integral using the FTC

(a) $\int_0^2 3x^2\,dx = x^3 \Big|_0^2 = 2^3 - 0^3 = 8 - 0 = 8$

(b) $\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx$ $= \sin x \Big|_{\frac{\pi}{2}}^{\pi}$

$= \sin \pi - \sin \frac{\pi}{2} = 0 - 1 = -1$



Question

(c) $\int_1^9 \frac{1}{2} u^{-1/2}\, du$ $= u^{1/2}\Big|_1^9 = \sqrt{9} - \sqrt{1} = 3 - 1 = 2$

(a) 8

(b) $\frac{13}{54}$

(c) 2

(d) $-\frac{1}{3}$

(d) $\int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\, dt = \sin^{-1} t \Big|_0^{\frac{1}{2}}$

$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$

$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$

Caveat! The FTC doesn't apply if f is not continuous!

The function $f(x) = \frac{1}{x^2}$ is positive everywhere on its domain. Now consider the calculation

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

Is this believable? Why or why not?

No, the graph of f is always above the x-axis, so a negative integral is not possible.

An Observation

If f is differentiable on $[a, b]$, note that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This says that:

The integral of the **rate of change** of f over the interval $[a, b]$ is the **net change** of the function, $f(b) - f(a)$, over this interval.

Remember the example: the *area* under the velocity curve gave the net change in position!