July 5 Math 1190 sec. 51 Summer 2017

Section 5.3: The Fundamental Theorem of Calculus

FTC part 1: If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a, b)!
"FTC" = "fundamental theorem of calculus"

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

Notation: Once we find an antiderivative F, we usually write the process like

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example

Evaluate $\dfrac{d}{dx}\displaystyle\int_1^{\sqrt{x}} \cot(t)\,dt$

$$= \cot(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\cot(\sqrt{x})}{2\sqrt{x}}$$

Chain rule

outside

$$f(u) = \int_1^u \cot(t)\,dt$$

$$\text{FTC} \Rightarrow f'(u) = \cot(u)$$

inside

$$u = \sqrt{x} \qquad u' = \frac{1}{2\sqrt{x}}$$

Example

Evaluate $\displaystyle\int_{-1}^{1} \frac{dy}{1+y^2}$ $= \displaystyle\int_{-1}^{1} \frac{1}{1+y^2}\,dy$

$$= \tan^{-1} y \Big|_{-1}^{1}$$

$$= \tan^{-1} 1 - \tan^{-1}(-1) = \frac{\pi}{4} - \left(\frac{-\pi}{4}\right) = \frac{\pi}{2}$$

Question

$$\int_2^3 \frac{1}{x}\,dx = \ln|x| \Big|_2^3 = \ln|3| - \ln|2|$$

$$= \ln 3 - \ln 2$$

(a) $-\dfrac{1}{6}$

(b) $\dfrac{1}{6}$

(c) $\ln 3 - \ln 2$

(d) $\ln 1$

Connecting the parts of the FTC

Use the FTC part 2 (treat x as though it were a constant) to evaluate

$$g(x) = \int_0^{x^2} \sec^2(t)\,dt \quad = \tan t \Big|_0^{x^2}$$

$$= \tan(x^2) - \tan(0)$$

$$= \tan(x^2) - 0 \quad = \tan(x^2)$$

$$g(x) = \tan(x^2)$$

Connecting the parts of the FTC

Use the results that you obtained to find $g'(x)$ using derivative rules from earlier chapters.

$$g(x) = \tan(x^2)$$

Outside

$f(u) = \tan(u) \quad f'(u) = \sec^2 u$

Inside

$u = x^2 \quad u' = 2x$

$$g'(x) = \sec^2(x^2) \cdot (2x)$$

$$= 2x \sec^2(x^2)$$

Connecting the parts of the FTC

Now use the FTC part 1 to evaluate the derivative

$$\frac{d}{dx}\int_0^{x^2} \sec^2(t)\,dt$$

$= \sec^2(x^2)\cdot(2x)$

$= 2x\sec^2(x^2)$

outside

$f(u) = \int_0^u \sec^2 t\,dt$

$f'(u) = \sec^2(u)$

inside

$u = x^2 \qquad u' = 2x$

How does this compare to what you get using the old rules?

They're the same! (duh!)

Section 5.4: Properties of the Definite Integral

Suppose that f and g are integrable on $[a, b]$ and let k be constant.

I. $$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

II. $$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

II. $$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Examples

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

(i) $\displaystyle\int_1^4 -2f(x)\,dx$ $= -2\displaystyle\int_1^4 f(x)\,dx = -2(3) = -6$

(ii) $\displaystyle\int_1^4 [f(x)+3g(x)]\,dx$ $= \displaystyle\int_1^4 f(x)\,dx + \int_1^4 3g(x)\,dx$

$= \displaystyle\int_1^4 f(x)\,dx + 3\int_1^4 g(x)\,dx = 3+3(-7) = 3-21$

$= -18$

Question

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

$$\int_1^4 [g(x) - 3f(x)]\,dx \quad = \int_1^4 g(x)\,dx - 3\int_1^4 f(x)\,dx$$

$$= -7 - 3(3)$$

(a) 16

(b) -16

(c) -2

(d) 2

The Sum/Difference in General

If $f_1, f_2, \ldots, f_n$ are integrable on $[a, b]$ and $k_1, k_2, \ldots, k_n$ are constants, then

$$\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\, dx =$$

$$k_1 \int_a^b f_1(x)\, dx + k_2 \int_a^b f_2(x)\, dx + \cdots + k_n \int_a^b f_n(x)\, dx$$

Example

Evaluate $\displaystyle\int_1^2 \frac{x^3+2x^2+4}{x}\,dx$

we have to do the distribution first

$$= \int_1^2 \left(\frac{x^3}{x} + \frac{2x^2}{x} + \frac{4}{x} \right) dx$$

$$= \int_1^2 \left(x^2 + 2x + 4 \cdot \frac{1}{x} \right) dx$$

$$= \int_1^2 x^2\,dx + \int_1^2 2x\,dx + \int_1^2 4 \cdot \frac{1}{x}\,dx$$

$$= \int_1^2 x^2\,dx + \int_1^2 2x\,dx + 4\int_1^2 \frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\Big|_1^2 + x^2\Big|_1^2 + 4\ln|x|\Big|_1^2$$

$$= \frac{2^3}{3} - \frac{1^3}{3} + \left(2^2 - 1^2\right) + \left(4\ln|2| - 4\ln|1|\right)$$

$$= \frac{8}{3} - \frac{1}{3} + (4-1) + 4\ln 2$$

$$= \frac{7}{3} + 3 + 4\ln 2 = \frac{16}{3} + 4\ln 2$$

Properties of Definite Integrals Continued...

If f is integrable on any interval containing the numbers a, b, and c, then

$$\text{(IV)} \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example

Suppose f is integrable on $(-\infty, \infty)$. Suppose further that we know that

$$\int_3^9 f(x)\,dx = 4 \quad \text{and} \quad \int_5^9 f(x)\,dx = -3.$$

Evaluate $\displaystyle\int_3^5 f(x)\,dx = \int_3^9 f(x)\,dx + \int_9^5 f(x)\,dx$

$$= 4 + 3 = 7$$

$*$ Recall $\displaystyle\int_9^5 f(x)\,dx = -\int_5^9 f(x)\,dx = -(-3)$

Question

Suppose $\int_0^1 f(x)\,dx = 1$, $\int_1^2 f(x)\,dx = 2$, and $\int_2^3 f(x)\,dx = 3$. Then

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

(a) 0

(b) 6

(c) 4

(d) can't be determined without more information

Properties: Bounds on Integrals

(V) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

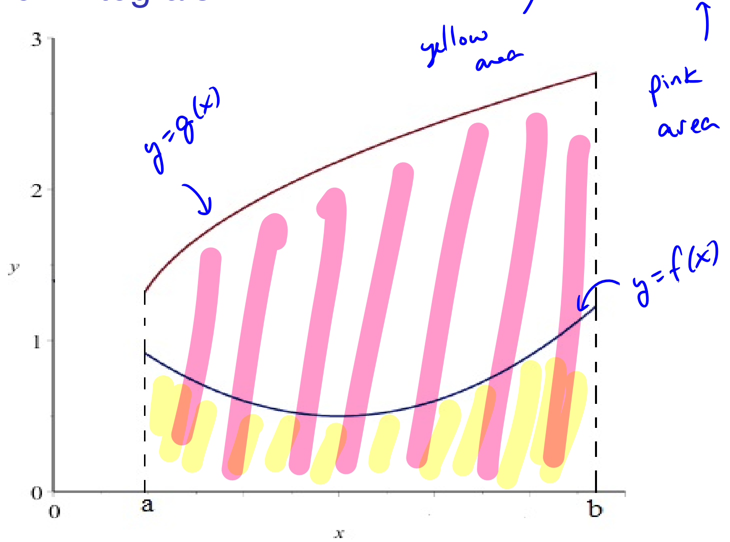
(VI) And, as an immediate consequence of (V), if $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

If f is continuous on $[a, b]$, we can take m to be the absolute minimum value and M the absolute maximum value of f as guaranteed by the Extreme Value Theorem.

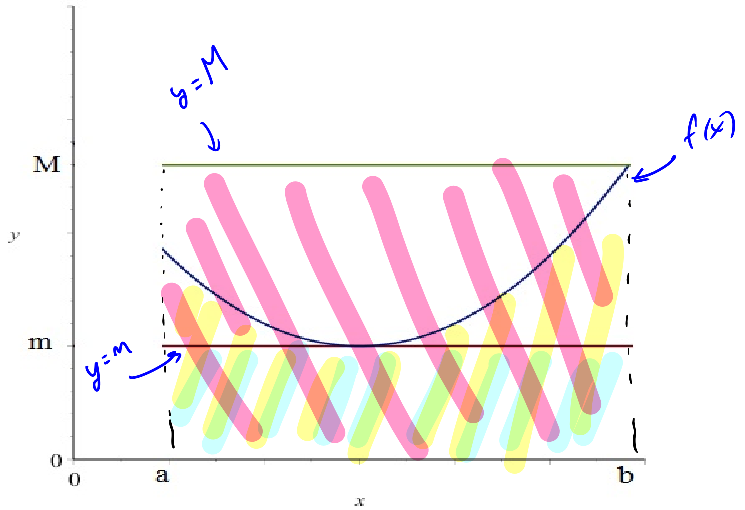
$f(x) \le g(x) \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Bounds on Integrals



Bounds on Integrals

$m \le f(x) \le M \Rightarrow m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$



Averages: The average value of a function

The average (arithmetic mean) of a collection of numbers $a_1, a_2, \ldots, a_n$ is

$$(a_1 + a_2 + \ldots + a_n)\frac{1}{n} = \sum_{i=1}^{n} a_i \frac{1}{n}$$

Can we define the average of infinitely many numbers?

How about the average value of some function f—i.e. the average of all of the numbers $f(x)$ for $a \leq x \leq b$?

Average value of a function

Start w/ f on $[a,b]$. Form an equally spaced partition

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b$$

$$\Delta x = \frac{b-a}{n} \implies \frac{1}{n} = \frac{\Delta x}{b-a}$$

Let $\{c_1, c_2, \ldots, c_n\}$ be any set of sample points.

The y's are $f(c_1), f(c_2), \ldots, f(c_n)$

The average of these is

$$\frac{f(c_1)+f(c_2)+\cdots+f(c_n)}{n} = \sum_{i=1}^{n} f(c_i)\frac{1}{n}$$

This gives a Riemann Sum if we use

$$\frac{1}{n} = \frac{\Delta x}{b-a} = \frac{1}{b-a} \Delta x$$

Avg for this partition is

$$\sum_{i=1}^{n} \left(f(c_i) \cdot \frac{1}{b-a} \Delta x \right)$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(c_i) \Delta x$$

we take the limit as $n \to \infty$ (over all possible partitions)

This gives an integral

$$\text{Average of } f = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Average Value of a Function and the Mean Value Theorem

Defintion: Let f be continuous on the closed interval $[a,b]$. Then the average value of f on $[a,b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Theorem: (The Mean Value Theorem for Integrals) If f is continuous on the interval $[a,b]$, then there exists a number u in $[a,b]$ such that

$$f(u) = f_{avg}, \quad \text{i.e.} \quad \int_a^b f(x)\,dx = f(u)(b-a).$$

Question

Find the average value of $f(x) = \sqrt{x}$ on the interval $[0,4]$. That is, compute

$$f_{avg} = \frac{1}{4-0}\int_0^4 x^{1/2}\,dx$$

$$= \frac{1}{4}\left[\frac{x^{3/2}}{3/2}\right|_0^4$$

$$= \frac{1}{4}\cdot\frac{2}{3}x^{3/2}\Big|_0^4$$

$$= \frac{1}{4}\left(\frac{2}{3}\cdot 4^{3/2} - \frac{2}{3}\cdot 0^{3/2}\right) = \frac{4}{3}$$

(a) $\frac{16}{3}$

(b) $\frac{4}{3}$

(c) $\frac{1}{2}$

(d) 2

Question

Find the value of f guaranteed by the MVT for integrals for $f(x) = \sqrt{x}$ on the interval $[0, 4]$. That is, find u such that

$$f(u) = f_{avg} = \frac{1}{4}\int_0^4 x^{1/2}\,dx = \frac{4}{3}$$

(a) $\sqrt{\frac{4}{3}}$

(b) $\frac{2}{\sqrt{3}}$

(c) $\frac{16}{9}$

(d) $\frac{16}{3}$

$f(u) = \frac{4}{3}$

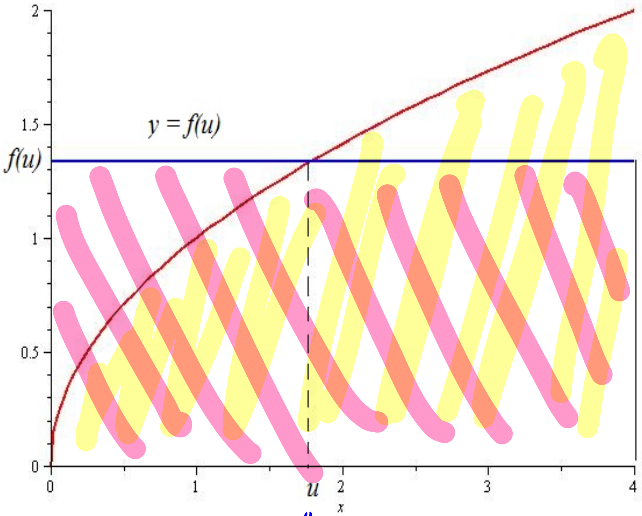
$\sqrt{u} = \frac{4}{3}$ square

$u = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$

MVT for Integrals Example

$\int_0^4 \sqrt{x}\,dx = f(u)(4-0)$

$\frac{4}{3} =$



$\frac{16}{9}$ "

Evaluate Each Integral

(a) $\displaystyle\int_2^1 (t+1)^2\,dt$

use $(t+1)^2 = t^2+2t+1$

$$= \int_2^1 (t^2+2t+1)\,dt = \frac{t^3}{3} + t^2 + t \Big|_2^1$$

$$= \frac{1^3}{3} + 1^2 + 1 - \left(\frac{2^3}{3} + 2^2 + 2\right)$$

$$= \frac{1}{3} + \cancel{1} + \cancel{1} - \frac{8}{3} - 4 - \cancel{2}$$

$$= -\frac{7}{3} - 4 = \frac{-19}{3}$$

Question

(b) $\int_1^3 x(3x+2)\,dx$ $=$ $\int_1^3 (3x^2+2x)\,dx$

(a) 86

(b) 34

(c) 47

(d) 28

(c) $\int_0^{\pi/4} \tan^2\theta \, d\theta$

we need the Trig ID

$\tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow \tan^2\theta = \sec^2\theta - 1$

$= \int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta$

$= \tan\theta - \theta \Big|_0^{\pi/4}$

$= \tan\frac{\pi}{4} - \frac{\pi}{4} - \left(\tan 0 - 0\right) = 1 - \frac{\pi}{4}$

Question

(d) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x}$ $= \displaystyle\int_{\pi/4}^{\pi/2} \csc^2 x \, dx = -\cot x \Big|_{\pi/4}^{\pi/2}$

$= -\cot\frac{\pi}{2} - \left(-\cot \pi/4\right)$

$= 0 - (-1) = 1$

(a) 1

(b) -1

(c) $\frac{1}{1-\frac{1}{\sqrt{2}}}$

(d) It is undefined since $\cos(\pi/2) = 0$.

Section 3.4: Newton's Method

We wish to find a number α that is a zero of the function $f(x)$

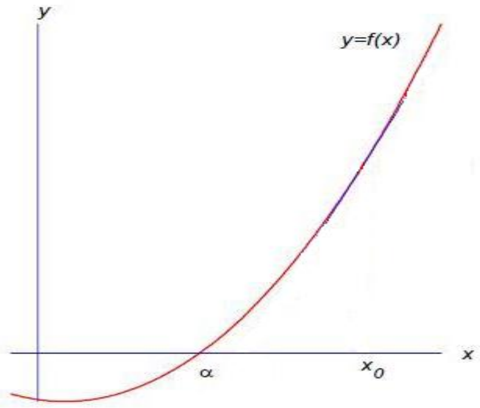


Figure: We begin by making a guess x_0 with the hope that $\alpha \approx x_0$.

Newton's Method

Next, we obtain a better approximation x_1 to the true root α.

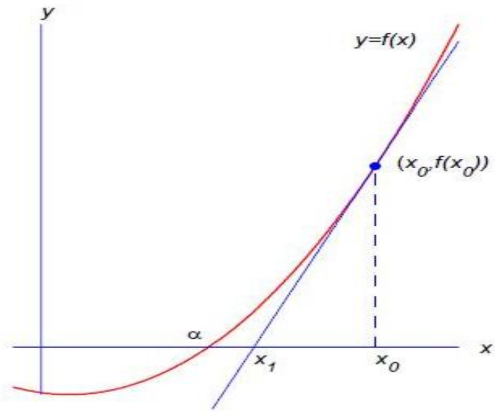


Figure: We choose x_1 to be the zero of $L(x)$, the tangent line approximation to f at x_0.

Formula for x_1:

We assume that $f(x)$ is differentiable on an interval containing α.

Start w/ guess x_0. Need the tangent line.

point $(x_0, f(x_0))$, slope $m = f'(x_0)$

calling y, $L(x)$

$$L(x) - f(x_0) = f'(x_0)(x - x_0)$$

$$y - y_0 = m(x - x_0)$$

The tangent line is

$$L(x) = f'(x_0)(x - x_0) + f(x_0)$$

This crosses the x-axis @ $(x_1, 0)$. This means

$$L(x_1) = 0.$$

$$0 = f'(x_0)(x_1 - x_0) + f(x_0)$$ solve for x_1

assume $f'(x_0) \neq 0$

$$-f'(x_0)(x_1 - x_0) = f(x_0)$$

$$x_1 - x_0 = \frac{f(x_0)}{-f'(x_0)} = -\frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Iterative Scheme for Newton's Method

We start with a *guess* x_0. Then set

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, we can find a tangent to the graph of f at $(x_1, f(x_1))$ and update again

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Newton's Iteration Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \ldots$$

The sequence begins with a starting *guess* x_0 expected to be near the desired root.

Exit Strategy for Newton's Method

Newton's method may or may not converge on the solution α. Since we hope that x_n is getting closer and closer to α, we generally stop when either

$$|x_{n+1} - x_n| < \text{Error Tol.}$$

or when

$$n \geq N$$

where "Error Tol." is some error tolerance and N is some predetermined maximum number of iterations.

If the latter condition is used to stop the process, the method is probably not working.

Example

Consider finding the real solution α of the equation

$$x^3 = x^2 + x + 1.$$

(a) Define an appropriate function $f(x)$ that has α as a root.

Let $f(x) = x^3 - x^2 - x - 1$

If $f(\alpha) = 0$ then $0 = \alpha^3 - \alpha^2 - \alpha - 1$

$\Rightarrow \quad \alpha^3 = \alpha^2 + \alpha + 1$

Example: $x^3 = x^2 + x + 1$

(b) Determine the Newton Iteration formula for this problem.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 1, 2, 3, \ldots$$

$$f(x) = x^3 - x^2 - x - 1, \qquad f'(x) = 3x^2 - 2x - 1$$

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - x_n - 1}{3x_n^2 - 2x_n - 1}$$

Example: $x^3 = x^2 + x + 1$

(c) Take $x_0 = 2$ and compute x_1.

$$x_1 = x_0 - \frac{x_0^3 - x_0^2 - x_0 - 1}{3x_0^2 - 2x_0 - 1}$$

$$= 2 - \frac{2^3 - 2^2 - 2 - 1}{3(2^2) - 2(2) - 1}$$

$$= 2 - \frac{1}{7} = \frac{14}{7} - \frac{1}{7} = \frac{13}{7}$$

Example: $x^3 = x^2 + x + 1$ TI-89

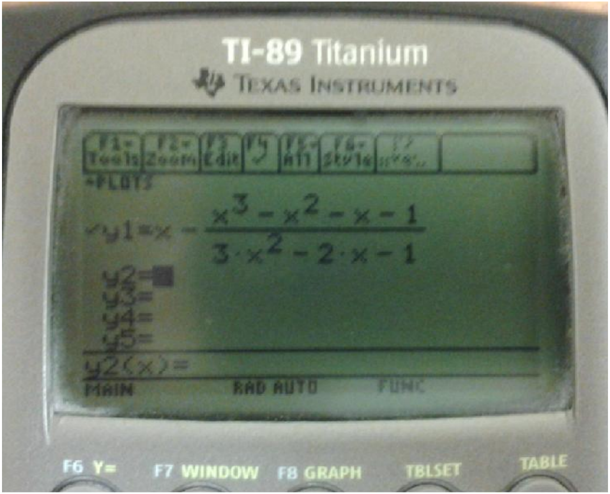


Figure: From the home window 2 [sto] x [enter], y1(x) [sto] x [enter], repeat.

Example: $x^3 = x^2 + x + 1$ TI-84

To access variables Y_i, hit [vars], select [Y-VARS], select [Function..], select desired variable.

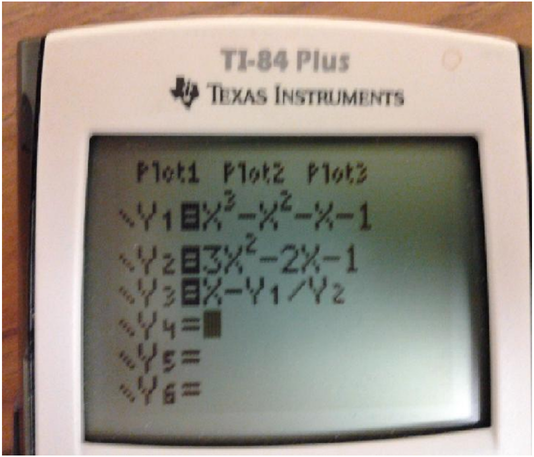


Figure: Set up $Y_1 = x^3 - x^2 - x - 1$, $Y_2 = 3x^2 - 2x - 1$ and $Y_3 = x - Y_1/Y_2$.

Example: $x^3 = x^2 + x + 1$ TI-84

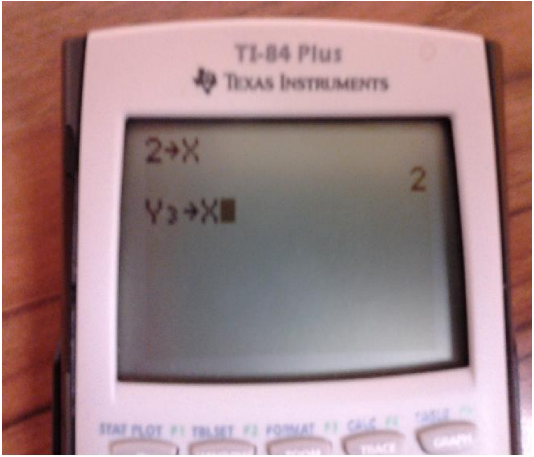


Figure: From the home screen 2 [sto] X [enter], then Y3 [sto] X [enter]. Keep hitting [enter].

Example: $x^3 = x^2 + x + 1$

Produced with Matlab with a tolerance of $\epsilon = 10^{-8}$.

n	x_n	$\lvert x_{n+1} - x_n \rvert$	$f(x_n)$
0	2.0000000000	0.1428571428	1.0000000000
1	1.8571428571	0.0175983436	0.0991253644
2	1.8395445134	0.0002577038	0.0014103289
3	1.8392868100	0.0000000548	0.0000003000
4	1.8392867552	0.0000000000	0.0000000000
5	1.8392867552		0.0000000000

Newton's method finds the root to within 10^{-8} in 5 full iterations. Another method called *bisection*, based on the Intermediate Value Theorem, requires 27 iterations when the initial assumption is that the root is between 1 and 2.

Computing Reciprocals without Division

Early computers (and even some supercomputers used today) did not compute with the operation $\div$. We consider a method for producing a reciprocal

$$\frac{1}{b} \quad \text{for a known nonzero number} \quad b$$

that relies only on the operations $+$, $-$, and $\times$.

Let $f(x) = b - \frac{1}{x}$. Then f is continuously differentiable for $x > 0$ and

$$f\left(\frac{1}{b}\right) = 0 \quad \text{i.e.} \quad \alpha = \frac{1}{b}$$

is the unique zero of f.

Example: Computing Reciprocal

Find the Newton's method iteration formula for solving $f(x) = 0$ where $f(x) = b - \frac{1}{x}$ and $b > 0$ is some constant.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = b - \frac{1}{x} = b - x^{-1}, \quad f'(x) = -(-x^{-2}) = x^{-2} = \frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{b - \frac{1}{x_n}}{\frac{1}{x_n^2}}$$

$$x_{n+1} = x_n - \left(b - \frac{1}{x_n}\right) \cdot \frac{x_n^2}{1}$$

$$= x_n - \left(b x_n^2 - \frac{1}{x_n} x_n^2\right)$$

$$= x_n - \left(b x_n^2 - x_n\right)$$

$$= x_n - b x_n^2 + x_n$$

$$\boxed{x_{n+1} = 2x_n - b x_n^2}$$

Example: Computing Reciprocal

It can be shown that the method will only find the reciprocal $\frac{1}{b}$ if the initial guess x_0 is close enough. In particular, it will only work if

$$0 < x_0 < \frac{2}{b}.$$

Example: Computing Reciprocal

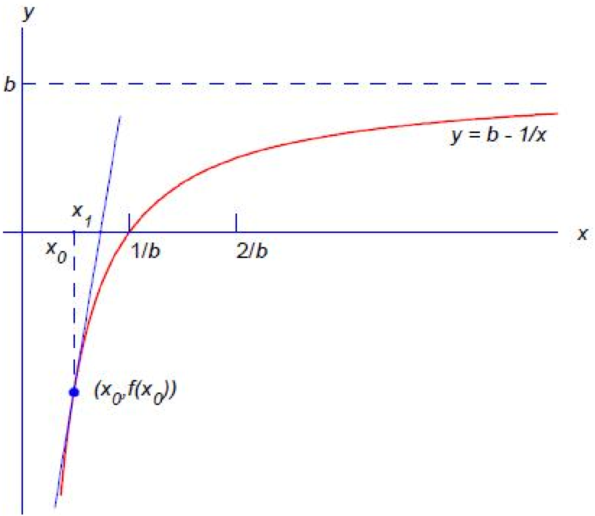


Figure: Illustration of using Newton's method to compute the reciprocal $1/b$.

Computing $\frac{1}{e}$

Start with an initial guess of $x_0 = 0.5$ and compute x_1.

Here $b = e$

$$x_{n+1} = 2x_n - e x_n^2$$

$$x_1 = 2x_0 - e x_0^2$$

$$x_1 = 2(0.5) - e(0.5)^2 = 1 - 0.25e$$

Example: Computing Reciprocal

Computing the reciprocal of the number e.

n	x_n	$\lvert x_{n+1} - x_n \rvert$	$f(x_n)$
0	0.5000	0.1796	0.7183
1	0.3204	0.0413	-0.4025
2	0.3618	0.0060	-0.0460
3	0.3678	0.0001	-0.0008
4	0.3679	0.0000	-0.0000
5	0.3679	0.0000	-0.0000
6	0.3679		0.0000

Six iterations are required with an initial guess of $x_0 = 0.5$ and a tolerance of $\epsilon = 10^{-8}$.

Example: Computing Reciprocal

Computing the reciprocal of the number e.

n	x_n	$\lvert x_{n+1} - x_n \rvert$	$f(x_n)$
0	0.7500	0.7790	1.3849
1	-0.0290	0.0313	37.1612
2	-0.0604	0.0703	19.2860
3	-0.1306	0.1770	10.3741
4	-0.3076	0.5648	5.9691
5	-0.8725	2.9416	3.8645
6	-3.8141	43.3572	2.9805

The same six iterations with an initial guess of $x_0 = 0.75$ produces garbage results.