July 6 Math 2254 sec 001 Summer 2015

Section 8.1: Sequences

Boundedness

Definition: A sequence $\{a_n\}$ is **bounded above** if there exists a number M such that

$$a_n \le M \quad \text{for all} \quad n \ge 1.$$

A sequence $\{a_n\}$ is **bounded below** if there exists a number m such that

$$a_n \ge m \quad \text{for all} \quad n \ge 1.$$

A sequence that is both bounded above and bounded below is called a **bounded sequence.**

Example

Determine if the sequence is bounded above, bounded below, and/or is a bounded sequence.

(a) $\{2^n\}_{n\geq 1}$ $2, 4, 8, 16, \ldots$

It's not bounded above.

It is bounded below since $2^n \geq 2$ for all n

It is not a bounded sequence.

(b) $\{1+(-1)^n\}_{n\geq 1}$ $0, 2, 0, 2, 0, 2, \ldots$

This is bounded above since $1+(-1)^n \leq 2$

This is bounded below since $1+(-1)^n \geq 0$

It is a bounded sequence.

Example continued...

Determine if the sequence is bounded above, bounded below, and/or is a bounded sequence.

(c) $\{c_n\}_{n\geq 1}$ where $c_n = \begin{cases} \frac{3}{n+2}, & n \text{ is even} \\ -4n, & n \text{ is odd} \end{cases}$

$c_1 = -4, \quad c_2 = \frac{3}{4}, \quad c_3 = -12, \quad c_4 = \frac{3}{6}, \quad c_5 = -20, \ldots$

This is not bounded below since the odd terms tend to $-\infty$.

It is bounded above since $c_n \leq \frac{3}{4}$

It's not a bounded sequence.

Theorems on Bounded and Monotonic Sequences

Theorem 1: If $\{s_n\}$ converges, then $\{s_n\}$ is bounded.

Theorem 2: If $\{s_n\}$ is nondecreasing and bounded above, then $\{s_n\}$ converges.

Theorem 3: If $\{s_n\}$ is nonincreasing and bounded below, then $\{s_n\}$ converges.

And the Grandaddy of them all...

The Monotonic Sequence Theorem: Every bounded monotonic sequence is convergent.

Example: Consider the sequence given by

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \quad \cdots \quad a_n = \sqrt{2a_{n-1}}.$$

It can be shown that

(1) a_n is strictly increasing, and (2) that $1 \le a_n \le 3$ for every n.

Discuss the convergence or divergence of $\{a_n\}$. If convergent, find its limit.

Property 1 says the sequence is monotonic.

Property 2 says it's bounded.

By the Monotonic Sequence Theorem, it is convergent.

Let's find its limit. Call the limit L.

$$a_n = \sqrt{2\, a_{n-1}}$$

Take $n \to \infty$, so $a_n \to L$ and $a_{n-1} \to L$.

Hence
$$L = \sqrt{2L}$$

square $L^2 = 2L \Rightarrow L^2 - 2L = 0$

$$L(L-2) = 0$$

Hence $L=0$ or $L=2$. (Reject 0 since $a_n \geq \sqrt{2}$)

Hence $L = 2$.

Section 8.2: Series

Definition: Suppose we have an infinite sequence of numbers $\{a_1, a_2, \dots\}$. We can consider summing them to form the expression

$$a_1 + a_2 + \cdots + a_n + \cdots$$

Such an expression is called a **series.** We may call it an **infinite series** to highlight that there are infinitely many summands.

Notation: We'll denote sums using a capital sigma (Greek letter "S") as follows:

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k.$$

If the limits, starting from $k = 1$ and going to ∞, are understood, we may simply write $\sum a_k$.

Sigma Notation

Capital Sigma

last index value

Summands

$$\sum_{k=1}^{\infty} a_k$$

k is called an index

first index value

Examples:

Some series would obviously give rise to a sum that is an infinty–e.g. the series

$$1 + 2 + 3 + \cdots + n + \cdots$$

Others give a well defined, finite sum inspite of there being infinitely many term. For example, it can be shown that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1.$$

Partial Sums

Definition: Let $\sum a_k$ be a series. The **sequence of partial sums** is the sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k. \end{aligned}$$

Example: For the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$, find the first three terms in the sequence of partial sums, s_1, s_2, and s_3. Here $a_k = \frac{1}{2^k}$

$s_1 = \frac{1}{2^1} = \frac{1}{2}$

$s_2 = \frac{1}{2^1} + \frac{1}{2^2} = \frac{3}{4}$

$s_3 = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{7}{8}$

Convergence or Divergence

Definition: Given a series $\sum a_k$, let $\{s_n\}$ denote the sequence of partial sums. If the sequence $\{s_n\}$ converges with limit s, that is

$$\text{if} \quad \lim_{n\to\infty} s_n = s,$$

then the series $\sum a_k$ is said to be **convergent**, and s is called the **sum** of the series. In this case, we write

$$\sum_{k=1}^{\infty} a_k = s.$$

If the sequence $\{s_n\}$ is divergent, then the series is said to be **divergent**.

Remark: A convergence or divergence of a series is defined in terms of the convergence or divergence of its sequence of partial sums.

Remark: If a sequence $\sum a_k$ converges, it is a **number**.

Example: $\sum_{k=1}^{\infty} \frac{1}{k^2+k}$ $= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots$

Use the partial fraction decomposition

$$\frac{1}{k^2+k} = \frac{1}{k} - \frac{1}{k+1}$$

to investigate the convergence or divergence of this series.

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

let $\{S_n\}$ be the sequence of partial sums

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

$$\vdots$$

$$S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

So $\quad S_n = 1 - \dfrac{1}{n+1} \quad$ for $\quad n \geq 1$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

The sequence $\{S_n\}$ converges with limit 1.

Hence the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k}$ converges

with sum 1.

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = 1$$

A Divergent Series

Use the well known result $1+2+\cdots+n=\frac{n(n+1)}{2}$ to investigate the convergence or divergence of the series

$$\sum_{k=1}^{\infty} k.$$

$$\sum_{k=1}^{\infty} k = 1+2+3+4+\cdots$$

Let $\{S_n\}$ be the sequence of partial sums.

$$S_1 = 1$$

$$S_2 = 1 + 2 = 3$$

$$S_3 = 1 + 2 + 3 = 6$$

$$\vdots$$

$$S_n = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Hence the sequence $\{S_n\}$ is divergent.

By definition, the series $\sum_{k=1}^{\infty} k$ is divergent.

Geometric Series

Let $a \neq 0$; the series

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

is called **geometric series.** The number r is called the **common ratio.**

Investigate the convergence or divergence of this series.

Recall * $\displaystyle\lim_{N\to\infty} r^N = \begin{cases} 0, & |r|<1 \\ \text{DNE}, & \text{if } r \le -1 \text{ or } r>1 \end{cases}$

Consider the $r=1$ case. If $r=1$, the series is $\displaystyle\sum_{n=0}^{\infty} a = a+a+a+a+\ldots$

Let $\{S_N\}$ be the sequence of partial sums.

$S_0 = a$

$S_1 = a + a = 2a$

$S_2 = a + a + a = 3a$

$S_N = a + a + \ldots + a$

$= (N+1)a$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} (N+1)a \quad \text{DNE}$$

If $r=1$, the series is divergent.

Suppose $r \neq 1$.

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$$

So $$S_0 = a \qquad S_N = a + ar + \ldots + ar^N$$

$$S_1 = a + ar$$

$$S_2 = a + ar + ar^2$$

$$S_N = a + ar + ar^2 + \ldots + ar^{N-1} + ar^N$$

$$rS_N = ar + ar^2 + ar^3 + \ldots + ar^N + ar^{N+1}$$

Subtract

$$S_N - rS_N = a - ar^{N+1}$$

$$(1-r)S_N = a(1-r^{N+1})$$

Note $1-r \neq 0$

$$S_N = \frac{a(1-r^{N+1})}{1-r}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a}{1-r}\left(1-r^{N+1}\right) = \begin{cases} \text{DNE} & \text{if } r>1 \\ & \text{or } r \le -1 \\ \dfrac{a}{1-r} & \text{if } |r|<1 \end{cases}$$

Geometric Series

Theorem: The series $a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$ is convergent if $|r| < 1$. In this case,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad |r| < 1.$$

If $|r| \geq 1$, the series is divergent.

Examples:

Determine the convergence or divergence of the geometric series. If convergent, find the sum.

(a) $2+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots$

Find the ratio r

$r = \frac{\frac{1}{2}}{2} = \frac{1}{4}$, $r = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{8}\cdot 2 = \frac{1}{4}$, $r = \frac{\frac{1}{32}}{\frac{1}{8}} = \frac{1}{4}$

$|r| = \left|\frac{1}{4}\right| < 1$, the series converges.

$a = 2$

$2+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\ldots = \frac{2}{1-\frac{1}{4}}\cdot\frac{4}{4} = \frac{8}{4-1} = \frac{8}{3}$

(b) $\sum_{n=0}^{\infty} \frac{5^{n+1}}{3^{2n-1}}$

$5^{n+1} = 5^n \cdot 5^1 = 5(5^n)$

$3^{2n-1} = 3^{2n} \cdot 3^{-1} = (3^2)^n \cdot 3^{-1}$

$= \frac{1}{3}(9^n)$

$= \sum_{n=0}^{\infty} \frac{5 \cdot 5^n}{\frac{1}{3} 9^n}$

$= \sum_{n=0}^{\infty} 15 \left(\frac{5}{9}\right)^n$

$a = 15$ and $r = \frac{5}{9}$

$|r| = \left|\frac{5}{9}\right| < 1$ the series converges.

$$\sum_{n=0}^{\infty} \frac{5^{n+1}}{3^{2n-1}} = \frac{15}{1-\frac{5}{9}} \cdot \frac{9}{9} = \frac{135}{9-5}$$

$$= \frac{135}{4}$$

Using a Geometric Series

Use a geometric series to find a rational equivalent to the number

$0.919\overline{191}$ $= 0.91 + 0.0091 + 0.000091 + \dots$

$$= \frac{91}{100} + \frac{91}{100^2} + \frac{91}{100^3} + \dots$$

$$= \frac{91}{100} + \frac{91}{100}\cdot\frac{1}{100} + \frac{91}{100}\left(\frac{1}{100}\right)^2 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{91}{100}\left(\frac{1}{100}\right)^n$$

$r = \frac{1}{100}$ so $|r| < 1$

also $a = \frac{91}{100}$

$$0.91\overline{91} = \frac{\frac{91}{100}}{1 - \frac{1}{100}} \cdot \frac{100}{100}$$

$$= \frac{91}{100 - 1} = \frac{91}{99}$$