July 6 Math 2306 sec 52 Summer 2016

Section 11: Linear Mechanical Equations

Here we consider **linear mechanical systems** which will be represented by a spring-mass-damper-driving force system. We'll build up the level of complexity in stages considering

- Simple Harmonic Motion (spring force, but no damping or driving)
- Spring-mass-damper (spring force w/damping, no driving)
- Spring-mass-damper with driving

Position: Equilibrium

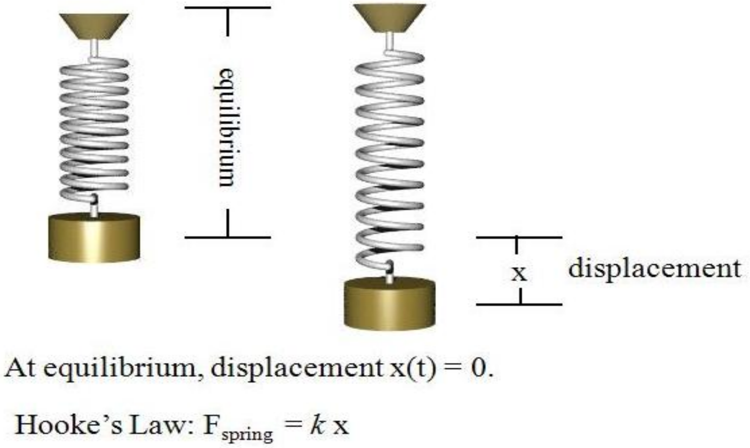

Figure: In the absence of any displacement, the system is at equilibrium. Displacement—i.e. **position**—$x(t)$, at time t, is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ Force = mass times acceleration

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ Force exerted by the spring is proportional to displacement
The force imparted by the spring opposes the direction of motion.

$$m\frac{d^2x}{dt^2} = -kx \quad \Longrightarrow \quad x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Convention We'll Use: Up will be positive ($x > 0$), and down will be negative ($x < 0$). This orientation is arbitrary and follows the convention in Trench.

Displacement *in Equilibrium*

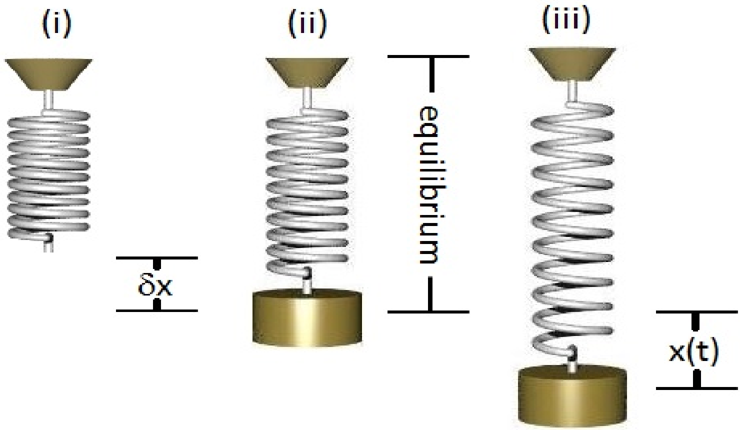

Figure: (i) a spring with no mass, (ii) a mass stretches the spring δx units **in equilibrium** to create a new equilibrium for the spring-mass system, (iii) displacement $x(t)$ is then measured from this new equilibrium position.

Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet in equilibrium, the by Hooke's law we compute the spring constant via the equation

$$W = k\delta x.$$

The units for k in this system of measure are lb/ft.

$$W \text{ lb} = k\,\delta x \text{ ft} \quad \Rightarrow \quad k = \frac{W}{\delta x}\ \frac{\text{lb}}{\text{ft}}$$

Obtaining the Mass (US Customary Units)

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

$$W \text{ lb} = m\, g \frac{\text{ft}}{\text{sec}^2} \implies m = \frac{W}{g} \frac{\text{lb}}{\text{ft/sec}^2} = \frac{W}{g} \text{ slugs}$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

Obtaining the Parameters (SI Units)

In SI units, the weight would be expressed in Newtons (N). The appropriate units for displacement would be meters (m). In these units, the spring constant would have units of N/m. Where again

$$k = \frac{W}{\delta x} \quad \text{where } \delta x \text{ is displacement in equilibrium.}$$

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\text{m}/\text{sec}^2.$$

Obtaining ω from *Displacment in Equilibrium* without Weight

If we know that an object displaces a spring δx units in equilibrium, then we can equate weight (mg) with spring force ($k\delta x$) by Hooke's law:

$$mg = k\delta x.$$

Without knowing the weight or the spring constant, the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively.

The characteristic eqn is (in the variable r)

$$r^2 + \omega^2 = 0 \Rightarrow r^2 = -\omega^2 \Rightarrow r = \pm\sqrt{-\omega^2} = \pm i\omega$$

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

$$x' = -\omega c_1 \sin(\omega t) + \omega c_2 \cos(\omega t)$$

$$x(0) = c_1 \cos(0) + c_2 \sin(0) = x_0 \Rightarrow c_1 = x_0$$

$$x'(0) = -\omega c_1 \sin(0) + \omega c_2 \cos(0) = x_1$$

$$\omega c_2 = x_1 \implies c_2 = \frac{x_1}{\omega}$$

The solution to the IVP is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Simple Harmonic Motion (no damping or driving)

The IVP governing displacement is

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{2}$$

called the **equation of motion***.

Caution: The phrase *equation of motion* is used differently by different authors. Some, including Trench, use this phrase to refer to the ODE of which (1) would be the example here. Others use it to refer to the **solution** to the associated IVP which is given by (2).

$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$*
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

*Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Example

An object stretches a spring 4 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

We know the equation looks like

$$x'' + \omega^2 x = 0$$

Given displacement in equilibrium, we can use

$$\omega^2 = \frac{g}{\delta x} \qquad \text{here } \delta x = 4 \text{ in} = \frac{1}{3} \text{ ft}.$$

$$\omega^2 = \frac{32 \ ft/sec^2}{1/3 \ ft} = 3 \cdot 32 \ \frac{1}{sec^2} = 96 \ \frac{1}{sec^2}$$

So the ODE is

$$x'' + 96x = 0$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

The form of the IVP is $\quad x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$

We're given a weight $W = 4$ lb.

To obtain the spring constant, we have

$$4\,\text{lb} = k\,\delta x = k\left(\tfrac{1}{2}\,\text{ft}\right) \qquad \delta x = 6\,\text{in} = \tfrac{1}{2}\,\text{ft}$$

$$\Longrightarrow \quad k = 8\,\text{lb}/\text{ft}$$

Also, from $W = mg$, $4\,lb = m(32\ ft/sec^2)$

$$\Rightarrow \quad m = \frac{4\ lb}{32\ ft/sec^2} = \frac{1}{8}\ \text{slugs}$$

So $\omega^2 = \frac{k}{m} = \frac{8}{1/8}\ \frac{1}{sec^2} = 64\ \frac{1}{sec^2}$

4 ft upward

$24\ \frac{ft}{sec}$ downward

Our IVP is

$$x'' + 64x = 0, \quad x(0) = 4 \quad x'(0) = -24$$

The char. eqn is $r^2 + 64 = 0$ $\quad r = \pm 8i$

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t) \qquad x(0) = c_1 = 4$$

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t) \qquad x'(0) = 8c_2 = -24$$

$$c_2 = -3$$

So

$$x(t) = 4\cos(8t) - 3\sin(8t)$$

Characteristics:

period $\quad T = \frac{2\pi}{8} = \frac{\pi}{4}$

frequency $\quad f = \frac{1}{T} = \frac{4}{\pi}$

Amplitude $A = \sqrt{(4)^2 + (-3)^2} = \sqrt{25} = 5$

Phase shift ϕ is defined by

$$\sin\phi = \frac{x_0}{A}, \quad \cos\phi = \frac{x_1}{\omega A}$$

so $\sin\phi = \frac{4}{5}$ and $\cos\phi = \frac{-3}{5}$

$\sin\phi > 0$ and $\cos\phi < 0$ so ϕ is a quadrant II angle

The smallest positive solution ϕ is

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21$$

about 127°

Side bar:

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\left[\frac{c_1}{A}\cos(\omega t) + \frac{c_2}{A}\sin(\omega t)\right]$$

Let $A = \sqrt{c_1^2 + c_2^2}$, then $\left(\frac{c_1}{A}\right)^2 + \left(\frac{c_2}{A}\right)^2 = \dfrac{c_1^2 + c_2^2}{c_1^2 + c_2^2} = 1$

Let $\frac{c_1}{A} = \sin\phi$ and $\frac{c_2}{A} = \cos\phi$

The right side becomes

$$A\left[\sin\phi\cos(\omega t) + \cos\phi\sin(\omega t)\right] = A\sin(\phi + \omega t)$$

Initial Conditions

To determine the future position, initial position and velocity must be given.

- In this course, we'll use the convention that up is positive and down is negative.

- If we're told that an object starts **at equilibrium**, then $x(0) = 0$.

- If we're told that an object starts **from rest**, then this means that $x'(0) = 0$.

Free Damped Motion

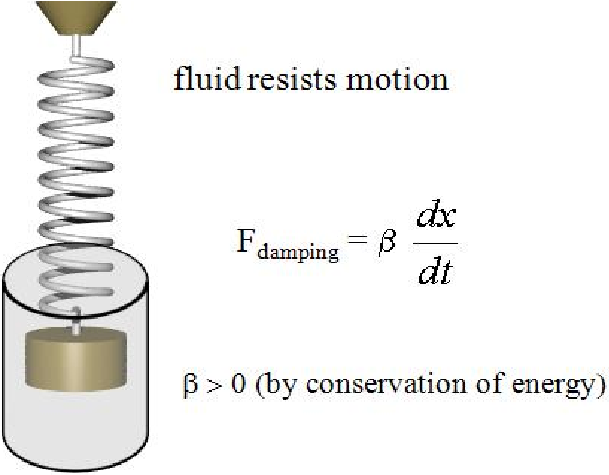

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

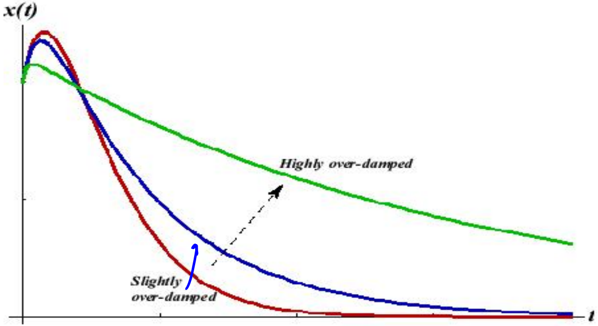

Figure: **(The red curve is "critical damping" and is only shown as a reference.)** Two distinct real roots. No oscillations. Approach to equilibrium may be slow. (Blue and Green curves are overdamped.)

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

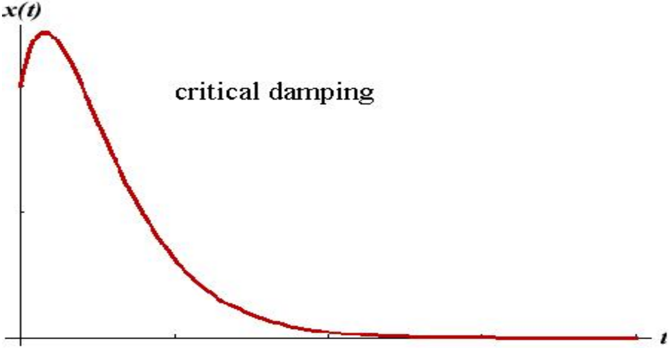

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

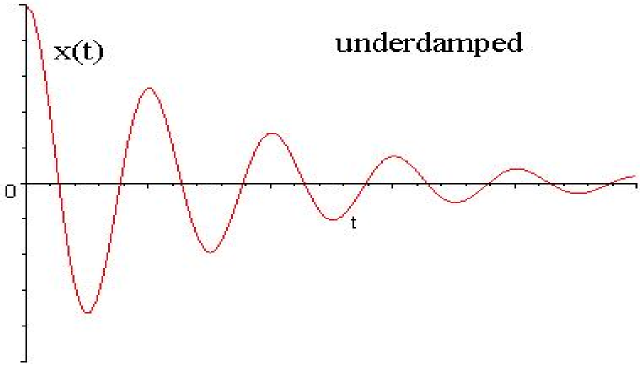

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

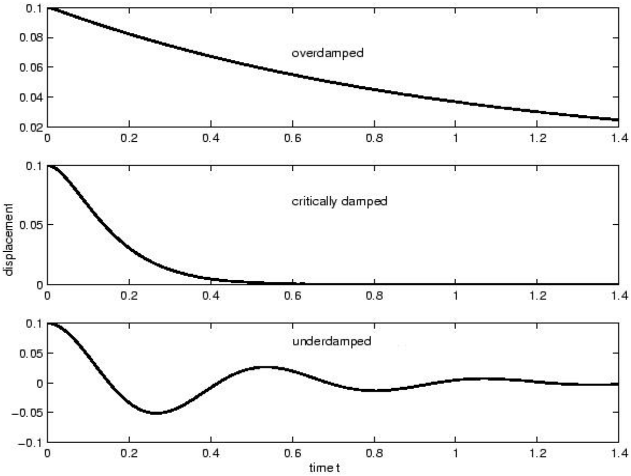

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The equation should look like $mx'' + \beta x' + kx = 0$

Given $m = 2$ kg, $k = 12$ N/m, and $\beta = 10\ \frac{kg}{sec}$

Our eqn is $2x'' + 10x' + 12x = 0$

In standard form $x'' + 5x' + 6x = 0$

The char. eqn is $r^2+5r+6=0$

$(r+2)(r+3)=0$ w/ roots $r=-2$ or $r=-3$

The system is over damped. (2 real roots)

Note $\lambda = \frac{\beta}{2m} = \frac{10}{2(2)} = \frac{5}{2}$ $\omega^2 = \frac{k}{m} = \frac{12}{2} = 6$

So $\lambda^2 - \omega^2 = \left(\frac{5}{2}\right)^2 - 6 = \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0$

so again the system is over damped.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

$mx'' + \beta x' + kx = 0$ given $m = 3$ kg, $k = 12$ N/m, $\beta = 12 \frac{kg}{sec}$.

Hence $3x'' + 12x' + 12x = 0$

In standard form $x'' + 4x' + 4x = 0$

The characteristic eqn is $r^2+4r+4=0$

$(r+2)^2=0 \Rightarrow r=-2$ repeated

The system is critically damped. (one real root)

From the givens, $x(0)=0$ and $x'(0)=1$

from equilibrium — upward velocity of 1 m/sec.

The general soln to the ODE is

$$x = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$x(0) = c_1 e^0 + c_2 \cdot 0 e^0 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$x'(0) = -2 \cdot 0 e^0 + c_2 e^0 - 2c_2 \cdot 0 e^0 = 1 \quad \Rightarrow \quad c_2 = 1$$

The solution to the IVP is

$$x = te^{-2t}$$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$x'' + \omega^2 x = F_0 \sin(\gamma t)$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$

If $\gamma \neq \omega$, this guess does not duplicate x_c, so this guess will work as written.

$x'' + \omega^2 x = F_0 \sin(\gamma t)$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ If $\gamma = \omega$, this duplicates the complementary solution. The correct form of the particular solution is

$$x_p = (A\cos(\gamma t) + B\sin(\gamma t)) \cdot t$$

$$= At\cos(\gamma t) + Bt\sin(\gamma t)$$

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega} \sin(\omega t) \right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t\cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

▸ Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Section 12: LRC Series Circuits

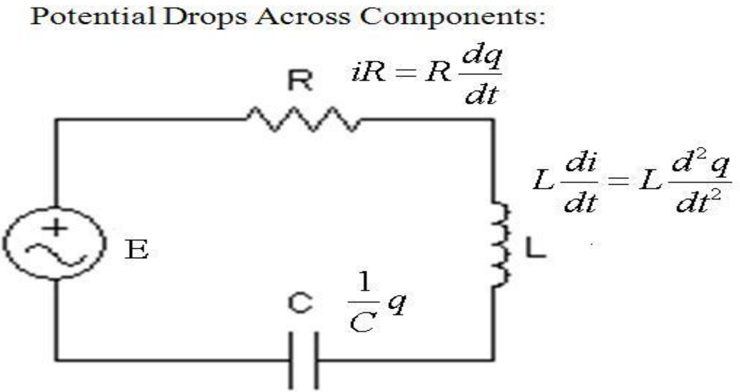

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

overdamped if	$R^2 - 4L/C > 0,$
critically damped if	$R^2 - 4L/C = 0,$
underdamped if	$R^2 - 4L/C < 0.$

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state charge of the system if the applied force is $E(t) = 5\cos(10t)$.

$$Lq'' + Rq' + \frac{1}{C}q = E \qquad \text{here; } L = \frac{1}{2}, \quad R = 10, \quad C = 4 \cdot 10^{-3}$$

$$\Rightarrow \quad \frac{1}{2}q'' + 10q' + \frac{1}{4 \cdot 10^{-3}}q = 5\cos(10t)$$

$$* \quad \frac{1}{4 \cdot 10^{-3}} = \frac{10^3}{4} = \frac{1000}{4} = 250$$

Standard form

$$q'' + 20q' + 500q = 10\cos(10t)$$

The char. eqn is $r^2 + 20r + 500 = 0$

with roots $r = -10 \pm 20i$

Using undetermined coefficients guess

$$q_p = A\cos(10t) + B\sin(10t)$$

$$q_c = c_1 e^{-10t}\cos(20t) + c_2 e^{-10t}\sin(20t)$$ no duplication

$$q_p' = -10A\sin(10t) + 10B\cos(10t)$$

$$q_p'' = -100A\cos(10t) - 100B\sin(10t)$$

$$q_p'' + 20q_p' + 500q_p = 10\cos(10t)$$

$$-100A\cos(10t)-100B\sin(10t)+20\left[-10A\sin(10t)+10B\cos(10t)\right]$$

$$+300\left[A\cos(10t)+B\sin(10t)\right]=10\cos(10t)$$

We'll have to finish on Monday July 11.