July 7 Math 2254 sec 001 Summer 2015

Section 8.2: Series

A Special Series: The Harmonic Series
Definition: The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

is called the **harmonic series.**

Theorem: The harmonic series is divergent.

Let $\{S_k\}$ be the sequence of partial sums.

Consider $k = 1, 2, 4, 8, \ldots, 2^p, \ldots$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right)$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + 3\left(\frac{1}{2}\right)$

$S_{16} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{8} + \frac{1}{9} + \dots + \frac{1}{16}$

$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{16} = 1 + 4\left(\frac{1}{2}\right)$

$$S_{2^p} \geq 1 + p\left(\frac{1}{2}\right)$$

$$\lim_{p\to\infty} S_{2^p} \geq \lim_{p\to\infty}\left(1+\frac{p}{2}\right) = \infty$$

The sequence of partial sums diverges.

Hence the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Section 8.3: Properties of Series and the Integral Test

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Caution: The converse is NOT true!

The harmonic series shows this.

$\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

A Test for Divergence

Theorem: (The Divergence Test)[1] If

$$\lim_{n\to\infty} a_n \quad \text{does not exists, or} \qquad \lim_{n\to\infty} a_n \neq 0,$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Summary

- If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.
- If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
- If $\lim_{n\to\infty} a_n = 0$, the series **may converge or may diverge**.

[1] The Divergence Test is also known as the **n**th **Term Test**.

Example:

If possible, determine if the series is convergent or divergent. If it is not possible to determine if the series converges, explain why.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n+3}$

Apply the divergence test.

$$a_n = \frac{2n}{n+3}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{n+3} = \lim_{n\to\infty} \left(\frac{2n}{n+3}\right)\frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{3}{n}} = \frac{2}{1+0} = 2$$

$2 \neq 0$. The series diverges by the divergence test.

Examples continued...

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Apply the divergence test.

$$a_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

The test is inconclusive.

Theorem: Some Properties of Convergent Series

Theorem: Suppose $\sum a_n$ and $\sum b_n$ are convergent series with sums S and T, respectively. Then the series

$$\sum(a_k + b_k), \quad \sum(a_k - b_k), \quad \text{and} \quad \sum ca_k \;\text{ for constant } c$$

are convergent with sums

$$\sum(a_k + b_k) = S + T, \quad \sum(a_k - b_k) = S - T,$$

$$\text{and} \quad \sum ca_k = cS.$$

Another Property of Series

Theorem: Adding or removing a **finite** number of terms from a series does not affect convergence or divergence. It will affect the sum in the convergent case.

For example,

$$\text{If} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges, then} \quad \sum_{n=7}^{\infty} a_n \quad \text{converges.}$$

$$\text{If} \quad \sum_{n=5}^{\infty} a_n \quad \text{diverges, then} \quad \sum_{n=2}^{\infty} a_n \quad \text{diverges.}$$

A More General Theorem

Theorem: If $\{a_n\}$ and $\{b_n\}$ are sequences such that for some $n_0 \geq 1$

$$b_n = a_n, \quad \text{for all} \quad n \geq n_0$$

then both series $\sum a_n$ and $\sum b_n$ converge or both series diverge.

Note: If they both converge, they may have different sums.

Example

Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{4}{n(n+1)}+\frac{2}{5^{n-1}}\right) = \sum_{n=1}^{\infty}\frac{4}{n(n+1)} + \sum_{n=1}^{\infty}\frac{2}{5^{n-1}}$$

if these converge

$$= 4+\frac{5}{2} = \frac{8+5}{2} = \frac{13}{2}$$

$$\sum_{n=1}^{\infty}\frac{4}{n(n+1)} = 4\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 4(1) = 4$$

$$\sum_{n=1}^{\infty} 2\left(\frac{1}{5}\right)^{n-1} = \frac{2}{1-\frac{1}{5}} = \frac{2\cdot 5}{\left(1-\frac{1}{5}\right)5} = \frac{10}{5-1} = \frac{10}{4} = \frac{5}{2}$$

The Integral Test

Recall:

Integrals were defined in terms of sums—Riemann Sums—and there is a geometric way, relating to area between curves, to interpret them.

Note: A series can be related to areas too

$$a_1 + a_2 + \cdots = a_1 \cdot 1 + a_2 \cdot 1 + \cdots$$

if the numbers a_k are heights and all the widths are 1. Of course, this makes best sense when the numbers a_k are positive.

Context for this Section: We will restrict our attention <u>for the moment</u> to series of nonnegative terms.

Relating an Integral to a Series (divergent)

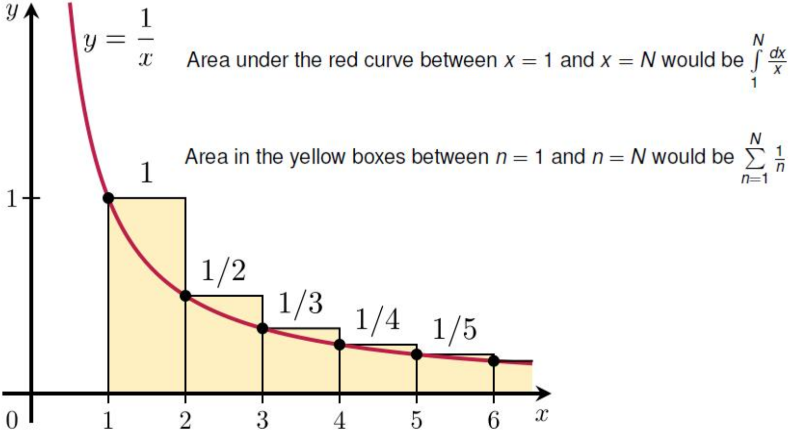

Figure: Comparison of *areas* related to $\int_1^\infty \frac{dx}{x}$ and $\sum_{n=1}^\infty \frac{1}{n}$.

Relating an Integral to a Series (convergent)

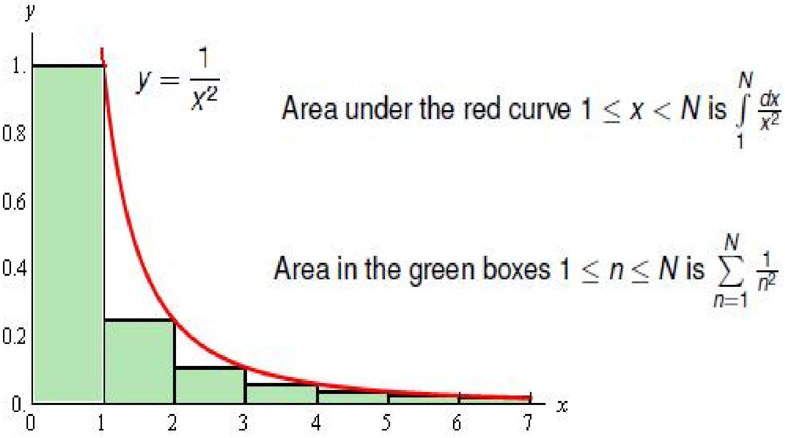

Figure: Comparison of *areas* related to $\int_1^\infty \frac{dx}{x^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Set Up for the Integral Test

Question: Does the series of positive terms $\sum_{n=1}^{\infty} a_n$ converge or diverge?

- Suppose f is a continuous, positive, decreasing function defined on the interval $[1, \infty)$.
- Also suppose that $a_n = f(n)$—the function $f(x)$ is the related function for the sequence $\{a_n\}$
- Assume that we are able to determine if the integral $\int_1^{\infty} f(x)\,dx$ converges or diverges.

Geometric Interpretation of the Integral Test

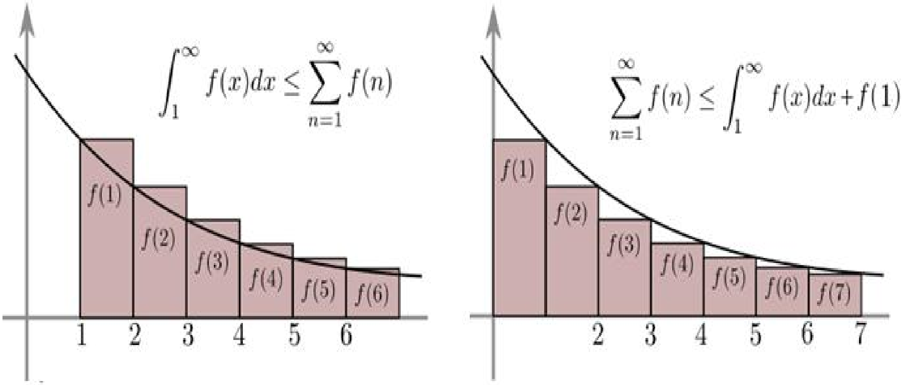

Figure: The possible value of the series can be trapped between the possible values of integrals.

The Integral Test

Theorem: Let $\sum a_n$ be a series of positive terms and let the function f defined on $[1, \infty)$ be continuous, positive and decreasing with

$$a_n = f(n).$$

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Both series and integral converge, or both series and integral diverge.

Examples:

Determine the convergence or divergence of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $f(x) = \dfrac{1}{x^2+1}$. f is positive, continuous, and decreasing.

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{dx}{x^2+1} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2+1}$$

$$= \lim_{t\to\infty} \tan^{-1}x \Big|_1^t$$

$$= \lim_{t\to\infty} \left(\tan^{-1}t - \tan^{-1}1\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

The integral converges.

Hence the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

converges by the integral test.

Examples:

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Let $f(x) = \frac{\ln x}{x}$, f is nonnegative and continuous.

$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} < 0$ for $x > e$

So f is decreasing for $x \geq 3$.

$\int_3^{\infty} f(x)\,dx = \int_3^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_3^{t} \frac{\ln x}{x}\,dx$ $*$

$$= \lim_{t\to\infty} \frac{1}{2}(\ln x)^2 \Big|_3^t = \lim_{t\to\infty}\left(\frac{1}{2}(\ln t)^2 - \frac{1}{2}(\ln 3)^2\right)$$

$$= \infty$$

The integral diverges.

The series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges by

the integral test. Since adding finitely many terms doesn't change convergence

$\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ also diverges.

$* \int \frac{\ln x}{x}\,dx \qquad \text{Let } u = \ln x \qquad du = \frac{1}{x}\,dx$

$$\int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

Special Series: p-series

Determine the values of p for which the series converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Recall

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{Converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$$

By the integral test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } p > 1 \text{ and}$$

$$\text{diverges if } p \le 1.$$

Special Series: p-series

$$\text{The series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is called a } \boldsymbol{p}\textbf{-series.}$$

Theorem: The p-series converges if $p > 1$ and diverges if $p \leq 1$.

Example: Determine if the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$

$p = \frac{5}{2} > 1$ it converges

$\frac{\sqrt{n}}{n^3} = \frac{1}{n^{3-\frac{1}{2}}} = \frac{1}{n^{5/2}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} 4 \frac{1}{n^{1/3}}$

$p = \frac{1}{3} \leq 1$

The series diverges.