July 9 Math 2254 sec 001 Summer 2015

Section 8.4: Comparison Tests

Note: In this section, we restrict our attention to series of nonnegative terms—i.e. $\sum a_n$ where $a_n \geq 0$ for each n.

We'll keep the following in mind:

- The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series. It converges if $p > 1$ and diverges if $p \leq 1$.

- A geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r| < 1$ and divergent if $|r| \geq 1$.

- If $\sum a_k$ has only nonnegative terms, then the sequence of partial sums $\{s_n\}$ must be a nondecreasing sequence—i.e. $s_{n+1} = s_n + a_{n+1} \geq s_n$.

Comparing Series: Motivating Example

Consider the two similar—yet different—series:

$$\text{(i)} \quad \sum_{n=0}^{\infty} \frac{1}{3^n} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{n=0}^{\infty} \frac{1}{3^n + 7}$$

Question: Does the series on the right, (ii), converge or diverge?

Comparing Series

$$\text{(i)} \quad \sum_{n=0}^{\infty} \frac{1}{3^n} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{n=0}^{\infty} \frac{1}{3^n+7}$$

- The one on the left, (i), is geometric $|r| = |1/3| < 1$—obviously convergent.
- The one on the right, (ii), is not geometric. And $f(x) = \frac{1}{3^x+7}$ isn't *easily* integrated.
- Does it help to notice that

$$\frac{1}{3^n+7} < \frac{1}{3^n}$$

for every value of n?

Comparing Sequences of Partial Sums

Let $s_k = \sum_{n=0}^{k} \frac{1}{3^n}$ and $t_k = \sum_{n=0}^{k} \frac{1}{3^n+7}$.

$s_0 = \frac{1}{3^0} = 1$ $\qquad t_0 = \frac{1}{3^0+7} = \frac{1}{8}$ $\qquad t_0 < s_0$

$s_1 = \frac{1}{3^0} + \frac{1}{3^1} = \frac{4}{3}$ $\qquad t_1 = \frac{1}{3^0+7} + \frac{1}{3^1+7} = \frac{1}{8} + \frac{1}{10} = \frac{9}{40}$

$t_1 < s_1$

$$\left.\begin{aligned} s_k &= 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^k} \\ t_k &= \frac{1}{1+7} + \frac{1}{3+7} + \frac{1}{3^2+7} + \dots + \frac{1}{3^k+7} \end{aligned}\right\} \Rightarrow t_k < s_k$$

for all $k \geq 0$

- $\lim_{k\to\infty} S_k = L$ is finite
- $\{t_k\}$ is a monotonic, increasing sequence.
- Since $t_k < S_k$ the sequence $\{t_k\}$ is bounded above.

Hence $\{t_k\}$ is convergent.

That is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+7}$ is a convergent series.

Comparing Series

What if we consider the two series

$$\text{(i)} \quad \sum_{n=3}^{\infty} \left(\frac{5}{2}\right)^n \qquad \text{and} \qquad \text{(ii)} \quad \sum_{n=3}^{\infty} \left[\ln(n)\left(\frac{5}{2}\right)^n\right]$$

- The one on the left, (i), is geometric $|r| = |5/2| > 1$—obviously divergent.
- The one on the right, (ii), is not geometric. And $f(x) = \ln(x)(5/2)^x$ isn't integrable (in terms of elementary functions)!
- And for every $n \geq 3$,

$$\ln(n)\left(\frac{5}{2}\right)^n > \left(\frac{5}{2}\right)^n.$$

Comparing Sequences of Partial Sums

Again, let $s_k = \sum_{n=3}^{k} \left(\frac{5}{2}\right)^n$ and $t_k = \sum_{n=3}^{k} \ln(n) \left(\frac{5}{2}\right)^n$.

$s_3 = \left(\frac{5}{2}\right)^3$ $\quad t_3 = (\ln 3)\left(\frac{5}{2}\right)^3$ $\quad t_3 > s_3$

$s_4 = \left(\frac{5}{2}\right)^3 + \left(\frac{5}{2}\right)^4$

$t_4 = (\ln 3)\left(\frac{5}{2}\right)^3 + (\ln 4)\left(\frac{5}{2}\right)^4$ $\quad t_4 > s_4$

in general $\quad t_k > s_k$ for $k \geq 3$

- $\lim_{k\to\infty} s_k = \infty$

Since $t_k > s_k$ for each k

$$\lim_{k\to\infty} t_k = \infty$$

So $\sum_{n=3}^{\infty}\left[\ln(n)\left(\frac{5}{2}\right)^n\right]$ is divergent.

Theorem: The Direct Comparison Test

Theorem: Suppose $\sum a_n$ and $\sum b_n$ are series of positive terms, such that[1]

$$a_n \leq b_n \quad \text{for each } n.$$

(i) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

(ii) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

[1] If the condition $a_n \leq b_n$ doesn't hold for some first finite number of terms, the result is unchanged. We could say $a_n \leq b_n$ for all $n \geq n_0$ for some number n_0.

Example

Determine the convergence or divergence of the series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+7}$

Since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges and

$\displaystyle\frac{1}{3^n+7} < \frac{1}{3^n}$ for all n

This series converges.

(b) $\displaystyle\sum_{n=3}^{\infty} \left[\ln(n)\left(\frac{5}{2}\right)^n\right]$

Since $\displaystyle\sum_{n=3}^{\infty} \left(\frac{5}{2}\right)^n$ diverges and

$\displaystyle\ln(n)\left(\frac{5}{2}\right)^n > \left(\frac{5}{2}\right)^n$ for $n \geq 3$

This series also diverges.

Example

Determine the convergence or divergence of the series.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^7 + 4n^2 + 3}}$

$$n^7 + 4n^2 + 3 \geq n^7 \Rightarrow$$

$$\sqrt[3]{n^7 + 4n^2 + 3} \geq \sqrt[3]{n^7} \Rightarrow$$

$$\frac{1}{\sqrt[3]{n^7 + 4n^2 + 3}} \leq \frac{1}{\sqrt[3]{n^7}} \Rightarrow$$

$$\frac{n}{\sqrt[3]{n^7 + 4n^2 + 3}} \leq \frac{n}{\sqrt[3]{n^7}}$$

$$\frac{n}{\sqrt[3]{n^7}} = \frac{n}{n^{7/3}} = \frac{1}{n^{7/3-1}} = \frac{1}{n^{4/3}}$$

We have $$\frac{n}{\sqrt[3]{n^7+4n^2+3}} \le \frac{1}{n^{4/3}}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is a p-series w/ $p = \frac{4}{3} > 1$

hence it converges.

By the direct comparison test
$\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^7+4n^2+3}}$ is convergent.

Note: as $n \to \infty$

$\sim$ symbol for this

$n^7 + 4n^2 + 3$ "behaves like" n^7

So $\dfrac{n}{\sqrt[3]{n^7+4n^2+3}}$ "behaves like" $\dfrac{n}{\sqrt[3]{n^7}} = \dfrac{1}{n^{4/3}}$

This indicates 2 things

① $\sum \frac{1}{n^{4/3}}$ is probably the series to use for comparison and

② Our series probably converges.

Example

Determine the convergence or divergence of the series.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4 - 2}$

As $n \to \infty$ $n^4 - 2 \sim n^4$

so $\dfrac{n^3}{n^4-2} \sim \dfrac{n^3}{n^4} = \dfrac{1}{n}$

$n^4 - 2 < n^4 \implies \dfrac{1}{n^4-2} > \dfrac{1}{n^4}$

$\implies \dfrac{n^3}{n^4-2} > \dfrac{n^3}{n^4} = \dfrac{1}{n}$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series. And for each $n \geq 2$

$$\frac{n^3}{n^4-2} > \frac{1}{n}.$$

Hence $\sum_{n=2}^{\infty} \frac{n^3}{n^4-2}$ diverges by the direct comparison test.

A Potential Fly in the Ointment

Consider the two series

$$\text{(i)} \quad \sum_{n=2}^{\infty} \frac{1}{3^n} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{n=2}^{\infty} \frac{1}{3^n - 7}$$

$$\text{Unfortunately} \quad \frac{1}{3^n - 7} \not\leq \frac{1}{3^n}. \quad \text{in fact} \quad \frac{1}{3^n} \leq \frac{1}{3^n - 7}.$$

Don't we strongly suspect that $\sum \frac{1}{3^n - 7}$ *ought* to be convergent? It is SO similar to $\sum \frac{1}{3^n}$.

We need a way to compare **similar** series that doesn't require such a specific inequality!

Theorem: The Limit Comparison Test

Theorem: Suppose $\sum a_n$ and $\sum b_n$ are series of positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \quad \text{and} \quad 0 < c < \infty,$$

Then either both series converge, or both series diverge.

Note: If the limit above fails to exist, or it is either 0 or ∞, then the test is inconclusive.

Example

Determine the convergence or divergence of the series.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

As $n\to\infty$ $n^2-1 \sim n^2$, so

$$n\sqrt{n^2-1} \sim n\sqrt{n^2} = n^2$$

That is, as $n\to\infty$ $\displaystyle\frac{1}{n\sqrt{n^2-1}} \sim \frac{1}{n^2}$

Let $\displaystyle a_n = \frac{1}{n\sqrt{n^2-1}}$ and $\displaystyle b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2-1}} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n}\sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^2}{n^2} - \frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = \frac{1}{\sqrt{1}} = 1$$

Since $0 < 1 < \infty$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent p-series, the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges by the limit comparison test.

Example

Determine the convergence or divergence of the series.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{5n^3 - 2n + 1}}$

as $n \to \infty$

$5n^3 - 2n + 1 \sim 5n^3$

$\sqrt[4]{5n^3 - 2n + 1} \sim \sqrt[4]{5n^3} = \sqrt[4]{5}\, n^{3/4}$

So $\displaystyle \frac{1}{\sqrt[4]{5n^3 - 2n + 1}} \sim \frac{1}{\sqrt[4]{5}\, n^{3/4}}$

Let $a_n = \dfrac{1}{n^{3/4}}$ and $b_n = \dfrac{1}{\sqrt[4]{5n^3 - 2n + 1}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{3/4}}}{\frac{1}{\sqrt[4]{5n^3 - 2n + 1}}} = \lim_{n\to\infty} \frac{\sqrt[4]{5n^3 - 2n + 1}}{n^{3/4}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[4]{5n^3-2n+1}}{\sqrt[4]{n^3}} = \lim_{n\to\infty} \sqrt[4]{\frac{5n^3-2n+1}{n^3}}$$

$$= \lim_{n\to\infty} \sqrt[4]{5-\frac{2}{n^2}+\frac{1}{n^3}} = \sqrt[4]{5-0+0} = \sqrt[4]{5}$$

Since $0<\sqrt[4]{5}<\infty$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$ is a divergent p-series, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{5n^3-2n+1}}$ diverges by the limit comparison test.

Analyzing Expressions with Powers and Roots

Identify the **leading term** in the numerator and in the denominator. Then take the ratio. For example:

$$\frac{\sqrt{3n^3+4n-6}}{\sqrt[5]{8n^{12}+32n^7-6n^4+12}}$$

as $n \to \infty$

$$\sqrt{3n^3+4n-6} \sim \sqrt{3n^3} = \sqrt{3}\, n^{3/2}$$

$$\sqrt[5]{8n^{12}+32n^7-6n^4+12} \sim \sqrt[5]{8n^{12}} = \sqrt[5]{8}\, n^{12/5}$$

$$\frac{n^{3/2}}{n^{12/5}} = \frac{1}{n^{\frac{12}{5}-\frac{3}{2}}} = \frac{1}{n^{\frac{24-15}{10}}} = \frac{1}{n^{9/10}}$$

So a series with these terms would

① be expected to diverge and

② can be compared to the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{9/10}}$$

Using a Comparison Test

- First try to determine if you think a series converges or diverges.
- Next, pick a series to compare it to such that **(1)** this series has the same convergence/divergence behavior, and **(2)** you can prove it! (usually use a p-series).
- Take the limit if using limit comparison—it doesn't matter who you call a_n and who you call b_n.
- Set up the inequality ($a_n \leq b_n$) if using direct comparison. If your series converges, it should be a_n. If your series diverges, you want it to be b_n.
- **Clearly state your final conclusion for all the world to see!**

Section 8.5: Alternating Series and Absolute Convergence

Definition: Let $\{a_n\}$ be a sequence of nonnegative numbers. A series of the form

$$\sum_{n=1}^{\infty}(-1)^n a_n, \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

is called an **alternating series**.

$$\sum_{n=1}^{\infty}(-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - a_5 + \dots$$

$$\sum_{n=1}^{\infty}(-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

Examples of Alternating Series

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is called the **alternating harmonic series.**

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{n+2} = -\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \cdots$$

is an alternating series.

Theorem: The Alternating Series Test

Theorem: Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ be an alternating series. If

$$(i) \quad \lim_{n\to\infty} a_n = 0$$

$$\text{and} \quad (ii) \quad a_{n+1} \le a_n \qquad \text{for all } n,$$

then the series is convergent.

Example

(a) Determine the convergence or divergence of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ so here $a_n = \frac{1}{n}$

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

(The first condition is true)

(ii) $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$

i.e. $a_{n+1} \leq a_n$

(The second condition is true)

Since both conditions are true, the alternating harmonic series converges by the alternating series test.

Example

Determine the convergence or divergence of the series

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+2}$ $= \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

So here, $a_n = \frac{n}{n+2}$.

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+2}$

$= \lim_{n\to\infty} \frac{n}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = \frac{1}{1+0} = 1$

(property (i) doesn't hold)

Use the divergence test.

$$\lim_{n\to\infty} (-1)^n \frac{n}{n+2} \quad \text{DNE}$$

(the even terms are tending to 1,
the odds to -1)

Hence $$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$$

diverges by the divergence test.