June 13 Math 2306 sec 52 Summer 2016

Section 5: First Order Equations Models and Applications

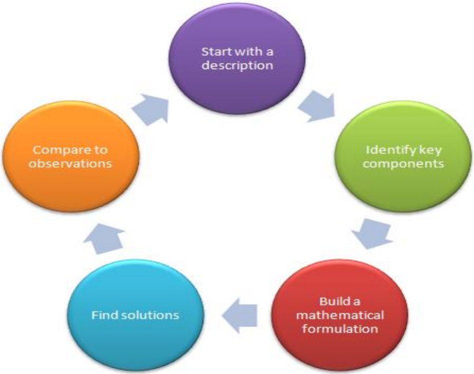


Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number† of rabbits expected in the population in 2021.

Variables: Let $P(t)$ be the population at time t.

$P \sim$ # of rabbits and $t \sim$ years.

$\frac{dP}{dt}$ = rate of change of population

We're told $\frac{dP}{dt} \propto P \implies \frac{dP}{dt} = kP$

for k - constant.

†We'll consider population density—i.e. number of individuals per unit space—when we say "number of rabbits."

We have $\frac{dP}{dt} = kP$ and we know $P = 58$ in 2011

and $P = 89$ in 2012.

Let $t = 0$ in 2011. Then $P(0) = 58$ and $P(1) = 89$.

Let's solve the IVP

$$\frac{dP}{dt} = kP \quad , \quad P(0) = 58$$

Separate variables:

$$\frac{1}{P}\frac{dP}{dt} = k$$

$$\frac{1}{P}\, dP = k\, dt$$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln|P| = kt + C$$

$$P = e^{kt+C} = e^{C}e^{kt}$$

$$P = Ae^{kt} \quad \text{when} \quad A = e^{C}$$

Apply $P(0) = 58$ $\quad 58 = Ae^{0} = A$

So $\quad P(t) = 58e^{kt}$ $\quad k$ - still unknown.

We can use $P(1) = 89$ to find k.

$$89 = P(1) = 58\, e^{k \cdot 1}$$

$$\Rightarrow \quad 58\, e^{k} = 89$$

$$e^{k} = \frac{89}{58} \quad \Rightarrow \quad k = \ln\left(\frac{89}{58}\right)$$

At time t, the population is

$$\boxed{P(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}}$$

In 2021, $t=10$.

$$P(10) = 58\,e^{10\ln\left(\frac{89}{58}\right)} \approx 4198$$

This model predicts 4198 rabbits in 2021.

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

The solution is $\quad P(t) = P(0)\, e^{kt}$

Series Circuits: RC-circuit

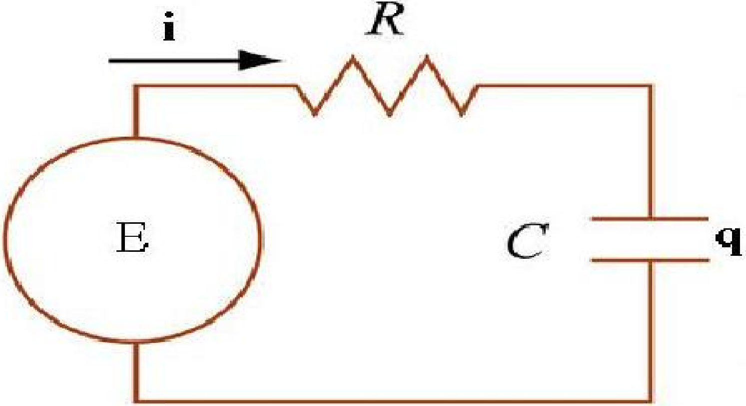


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

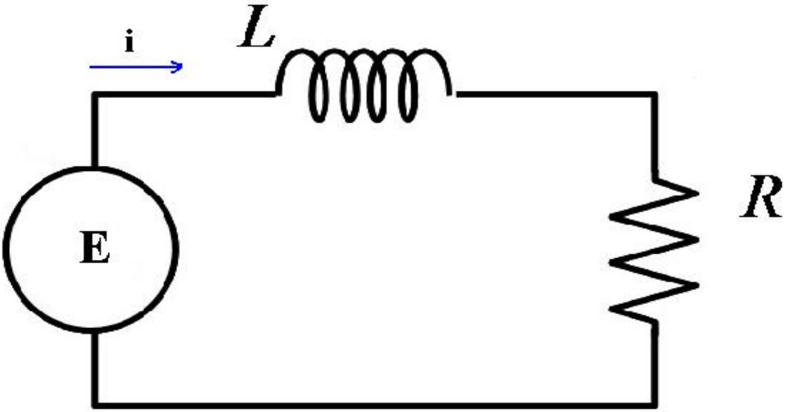


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω), Implied voltage E in volts (V),
Inductance L in henries (h), Charge q in coulombs (C),
Capacitance C in farads (f), Current i in amperes (A)

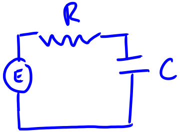
Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

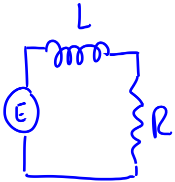


RC

$$Ri + \frac{1}{C}q = E \quad \text{as} \quad i = \frac{dq}{dt}$$

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

1st order linear ODE for charge q



$$L\frac{di}{dt} + Ri = E$$

1st order, linear ODE for current i.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

$$\Downarrow$$

$$1000\frac{dq}{dt} + \frac{1}{5\cdot 10^{-6}}q = 200$$

$E = 200$ volts

$R = 1000$ Ω

$C = 5\cdot 10^{-6}$ f

$q'(0) = i(0) = 0.4$ A

Note $\frac{1}{1000}\cdot\frac{1}{5\cdot 10^{-6}} = \frac{10^6}{5\cdot 10^3} = \frac{10^3}{5} = 200$

Standard form

$$\frac{dq}{dt} + 200q = \frac{1}{5} \quad , \quad q'(0) = 0.4$$

$P(t) = 200$. $\mu = e^{\int P(t)dt} = e^{\int 200 dt} = e^{200t}$

Hence $\frac{d}{dt}\left[e^{200t} q\right] = \frac{1}{5} e^{200t}$

$$\int \frac{d}{dt}\left[e^{200t} q\right] dt = \int \frac{1}{5} e^{200t} dt$$

$$e^{200t} q = \frac{1}{5 \cdot 200} e^{200t} + K$$

$$\Rightarrow \quad q = \frac{1}{1000} + k e^{-200t} , \qquad q'(t) = -200 k e^{-200t}$$

Apply $q'(0) = 0.4$ $\qquad 0.4 = -200 k e^{0} = -200k$

$$\Rightarrow \quad k = \frac{0.4}{-200} = -\frac{4}{10(200)} = \frac{.2}{1000} = \frac{-1}{500}$$

$$\boxed{\text{Finally, } \quad q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}}$$

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} \left(\frac{1}{1000} - \frac{1}{500} e^{-200t} \right) = \frac{1}{1000} - \frac{1}{500} \cdot 0 = \frac{1}{1000}$$

$$q \to \frac{1}{1000} \, C \quad \text{as} \quad t \to \infty .$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

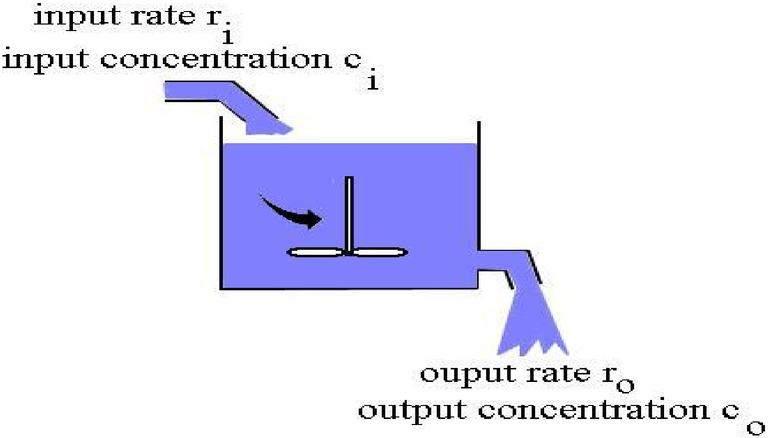


Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentration of the substance in the tank changes in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array}\right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i \cdot c_i.$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o \cdot c_o.$$

Building an Equation

c_o = concentration in the tank.

The concentration of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

$V(0)$ = initial fluid volume

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V(t)} A = r_i \cdot c_i$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$$\frac{dA}{dt} + \frac{r_o}{V}A = r_i c_i$$

$$r_i = 5\ \frac{\text{gal}}{\text{min}} \qquad r_o = 5\ \frac{\text{gal}}{\text{min}}$$

$$c_i = 2\ \text{lb}/\text{gal}$$

$$V = 500\,\text{gal} + \left(5\,\frac{\text{gal}}{\text{min}} - 5\,\frac{\text{gal}}{\text{min}}\right)t\ \text{min}$$

$$= 500\ \text{gal}$$

Initial condition $A(0) = 0$ (initially pure water)

$$\frac{dA}{dt} + \frac{5\,\frac{\text{gal}}{\text{min}}}{500\ \text{gal}} A = 5\,\frac{\text{gal}}{\text{min}} \cdot 2\,\frac{\text{lb}}{\text{gal}}$$

$$\frac{\text{lb}}{\text{min}} \rightarrow \frac{dA}{dt} + \frac{1}{100}\,\frac{1}{\text{min}} \overset{\text{lb}}{A} = 10\,\frac{\text{lb}}{\text{min}}$$

Solve the IVP $\frac{dA}{dt} + \frac{1}{100} A = 10 \qquad A(0) = 0$

$$P(t) = \frac{1}{100}, \quad \mu = e^{\int P(t)\,dt} = e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$$

so

$$\frac{d}{dt}\left[e^{\frac{1}{100}t} A\right] = 10\, e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left[e^{\frac{1}{100}t} A\right] dt = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$e^{\frac{1}{100}t} A = \frac{10}{1/100}\, e^{\frac{1}{100}t} + C$$

$$\Rightarrow \quad A(t) = 1000 + C e^{-\frac{1}{100}t}$$

Apply $A(0) = 0$

$$0 = 1000 + C e^{0} = 1000 + C \Rightarrow C = -1000$$

so the amount of salt

$$\boxed{A(t) = 1000 - 1000 e^{-\frac{1}{100}t}}$$

when $t=5$, the concentration is

$$C = \frac{A(5)}{V(5)} = \frac{1000 - 1000 e^{-\frac{1}{100}\cdot 5}}{500} \frac{\text{lb}}{\text{gal}} \approx 0.098 \frac{\text{lb}}{\text{gal}}$$