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Section 5: First Order Equations Models and Applications

We derived several mathematical models involving first order ODEs.

- Exponential growth ($k > 0$) or decay ($k < 0$): $\frac{dP}{dt} = kP$
- RC-series circuit for charge q: $R\frac{dq}{dt} + \frac{1}{C}q = E$
- LR-series circuit for current i: $L\frac{di}{dt} + Ri = E$
- Mixing problem for amount of some substance A:
  $\frac{dA}{dt} + \frac{r_o}{V(t)}A = r_i \cdot c_i$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity* M of the environment and the current population. Determine the differential equation satsified by P.

rate of change of P is $\frac{dP}{dt}$

$$\frac{dP}{dt} \propto P\,(M-P)$$

difference between carrying capacity and population

$$\Longrightarrow \quad \frac{dP}{dt} = kP(M-P) \quad \text{for some constant } k.$$

*The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation† and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

Let $P(0) = P_0$ Solve the separable eqn.

$$\frac{dP}{dt} = kP(M-P)$$

$$\frac{1}{P(M-P)}\,dP = k\,dt \Rightarrow \int \frac{1}{P(M-P)}\,dP = \int k\,dt$$

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†The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

$$\frac{1}{M}\int\left(\frac{1}{P}+\frac{1}{M-P}\right)dP=\int k\,dt$$

$$\int\left(\frac{1}{P}+\frac{1}{M-P}\right)dP=\int Mk\,dt$$

$\uparrow$ if $u=M-P$

$du=-dP$

$$\ln|P|-\ln|M-P|=Mkt+C$$

Solve for P

$$\ln\left|\frac{P}{M-P}\right|=Mkt+C$$

$$\left|\frac{P}{M-P}\right| = e^{Mkt+C} = e^{C}e^{Mkt}$$

Let $A = \pm e^{C}$

$$\frac{P}{M-P} = Ae^{Mkt} \Rightarrow P = Ae^{Mkt}(M-P)$$

$$P = AMe^{Mkt} - Ae^{Mkt}P \Rightarrow$$

$$P(1+Ae^{Mkt}) = AMe^{Mkt}$$

$$P = \frac{AMe^{Mkt}}{1+Ae^{Mkt}}$$

Now use $P(0) = P_0$ to solve for A.

From $\frac{P}{M-P} = Ae^{Mkt}$, $\frac{P_0}{M-P_0} = Ae^{0} = A$

Hence

$$P(t) = \frac{\frac{P_0}{M-P_0} M e^{Mkt}}{1 + \frac{P_0}{M-P_0} e^{Mkt}}$$

Clear fractions

$$P(t) = \left(\frac{\frac{P_0}{M-P_0} M e^{Mkt}}{1 + \frac{P_0}{M-P_0} e^{Mkt}} \right) \frac{M-P_0}{M-P_0}$$

$$\boxed{P(t) = \frac{P_0 M e^{Mkt}}{M - P_0 + P_0 e^{Mkt}}}$$

This is the population.

If $P_0 = 0$ then $P(t) = 0$ for all t.

If $P_0 \neq 0$

$Mk > 0$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0 M e^{Mkt}}{M - P_0 + P_0 e^{Mkt}} = \text{"}\frac{\infty}{\infty}\text{"}$$

Use l'Hospital's rule

$$= \lim_{t\to\infty} \frac{\cancel{P_0} M \cancel{M} \cancel{k} \cancel{e^{Mkt}}}{\cancel{P_0} \cancel{M} \cancel{k} \cancel{e^{Mkt}}}$$

$$= \lim_{t\to\infty} M = M$$

If the starting population is not zero, the population tends to the maximum the environment can handle.

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

to solve subject to conditions

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\quad \ldots,\quad y^{(n-1)}(x_0)=y_{n-1}.$$

The problem is called **homogeneous** if $g(x)\equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Example

Use only a little clever intuition to solve the IVP

$$y'' + 3y' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 0$$

This is homogenous w/ $a_2(x)=1$, $a_1(x)=3$, $a_0(x)=-2$.

All are continuous on $(-\infty,\infty)$ and $a_2(x)\neq 0$ for all x.

If $y(x)=0$ for all x, then $y(0)=0$ and $y'(0)=0$.

And $y''+3y'-2y = 0+3\cdot 0-2\cdot 0=0$.

By uniqueness, $y(x)=0$ is the only solution to the IVP.

A Second Order Linear Boundary Value Problem

consists of a problem

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x),\quad a<x<b$$

to solve subject to a pair of conditions[‡]

$$y(a)=y_0,\quad y(b)=y_1.$$

However similar this is in appearance, the existence and uniqueness result **does not hold** for this BVP!

[‡]Other conditions on y and/or y' can be imposed. The key characteristic is that conditions are imposed at both end points $x=a$ and $x=b$.

BVP Examples

All solutions of the ODE $y'' + 4y = 0$ are of the form

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

Solve the BVP

$$y'' + 4y = 0, \quad 0 < x < \frac{\pi}{4} \quad y(0) = 0, \quad y\left(\frac{\pi}{4}\right) = 0.$$

$y = c_1 \cos(2x) + c_2 \sin(2x)$

$y(0) = 0 \quad \Rightarrow \quad 0 = c_1 \cos(0) + c_2 \sin(0)$

$0 = c_1(1) + c_2(0) \quad \Rightarrow \quad c_1 = 0$

So $\quad y = C_2 \sin(2x)$

$$y(\pi/4) = 0 \implies \quad 0 = C_2 \sin\left(2 \cdot \frac{\pi}{4}\right)$$

$$0 = C_2(1) \implies C_2 = 0$$

Since $c_1 = 0$ and $c_2 = 0$, we get exactly one solution $\quad y = 0$.

BVP Examples

Solve the BVP

$$y'' + 4y = 0, \quad 0 < x < \pi \quad y(0) = 0, \quad y(\pi) = 0.$$

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

$y(0) = 0 \implies 0 = c_1 \cos(0) + c_2 \sin(0) \implies c_1 = 0$

$y(\pi) = 0 \implies 0 = c_2 \sin(2\pi) = c_2 \cdot 0$

$0 = 0$ is true for all c_2.

$y = c_2 \sin(2x)$ is a solution for any real number c_2.

The BVP has infinitely many solutions.

BVP Examples

Solve the BVP

$$y'' + 4y = 0, \quad 0 < x < \pi \quad y(0) = 0, \quad y(\pi) = 1.$$

$y = c_1 \cos(2x) + c_2 \sin(2x)$ and as before $y(0)=0 \Rightarrow c_1 = 0$.

$y(\pi) = 1 \Rightarrow \quad 1 = c_2 \sin(2\pi) = c_2 \cdot 0$

$1 = 0$ this is false for all c_2.

This BVP has no solutions.

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition.**

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a linear, homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \dots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Note: setting all c's to zero, we can always make the left side of the equation zero. If the only way to get zero is for all $c_i = 0$, the functions are linearly independent.

Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.

Can I find c_1, c_2, c_3 not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all } x \text{ in } (-\infty, \infty)?$$

$$c_1 \sin^2 x + c_2 \cos^2 x + c_3 \cdot 1 = 0$$

Since $\sin^2 x + \cos^2 x = 1$, take $c_1 = c_2$, $c_3 = -c_1$

For example if $c_1 = c_2 = 1$ and $c_3 = -1$

then at least one of them is nonzero,

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = \sin^2 x + \cos^2 x - 1 = 0$$

holds for all real x.

So $f_1(x), f_2(x), f_3(x)$ are linearly dependent.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Can we show that $c_1 f_1(x) + c_2 f_2(x) = 0$ for all x

is only possible if $c_1 = 0$ <u>and</u> $c_2 = 0$?

Consider $c_1 \sin x + c_2 \cos x = 0$ for all real x *

Since it's * true for all real x, it's true when $x = 0$.

$$c_1 \sin 0 + c_2 \cos 0 = 0$$

$$c_1(0) + c_2(1) = 0 \implies c_2 = 0$$

* Must hold when $x = \pi/2$. So

$$c_1 \sin\left(\frac{\pi}{2}\right) + 0 \cdot \cos(\pi/2) = 0$$

$$c_1 (1) = 0 \Rightarrow c_1 = 0$$

Both coefficients c_1 and c_2 must be zero.

$f_1(x)$ and $f_2(x)$ are linearly independent.

Suppose $c_1 \neq 0$ From $c_1 \sin x + c_2 \cos x = 0$

$$\Rightarrow \sin x = \frac{-c_2}{c_1} \cos x$$

$f_1(x)$ is just a constant multiple of $f_2(x)$. This isn't true for $\sin x$ and $\cos x$.

Determine if the set is Linearly Dependent or Independent

$I = (-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

Can we find c_1, c_2, c_3 not all zero such that

$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all real x

$$c_1(x^2) + c_2(4x) + c_3(x - x^2) = 0$$

$$c_1 x^2 + 4c_2 x + c_3 x - c_3 x^2 = 0$$

$$(c_1 - c_3)x^2 + (4c_2 + c_3)x = 0$$

The x^2 cancels if $c_1 = c_3$.

The x cancels if $c_3 = -4c_2$ i.e. $c_2 = \frac{-1}{4} c_3$

One example is $c_2 = 1$, $c_1 = -4$, $c_3 = -4$.

If $c_1 = -4$, $c_2 = 1$, $c_3 = -4$, then at least one is nonzero and $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all real x.

$f_1(x), f_2(x), f_3(x)$ are linearly dependent.

Definition of Wronskian

Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Recall Determinants

For a 2×2 matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Using a cofactor expansion (across the top row for illustration),

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

2 functions, we'll have a 2x2 matrix.

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} =$$

$$= \sin x\,(-\sin x) - \cos x\,(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\sin^2 x + \cos^2 x) = -1$$

$$W(f_1, f_2)(x) = -1$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions, we'll have a 3x3 matrix.

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1 - 2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2\Big(-8 - 0(1-2x)\Big) - 4x\Big(-4x - 2(1-2x)\Big) + (x-x^2)(0-8)$$

$$= -8x^2 - 4x(-4x - 2 + 4x) - 8x + 8x^2$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

$$W(f_1, f_2, f_3)(x) = 0$$

Theorem (a test for linear independence)

Let f_1, $f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If y_1, $y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[§] each x in I.

[§] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.