June 1 Math 2306 sec 52 Summer 2016

Section 1: Introduction Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$. Note that

$$\frac{d^2y}{dx^2} + 4y = 0.$$

$-4\cos(2x)$ $\qquad$ $4\cos(2x)$

Differential Equation

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a **differential equation.**

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it? Also, is $\cos(2x)$ the only possible function that y could be?

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Independent Variable: will appear as one that derivatives are taken with respect to.
Dependent Variable: will appear as one that derivatives are taken of.

$$\frac{dy}{dx} \quad \leftarrow \text{dependent } y \qquad \frac{du}{dt} \quad \leftarrow \text{dep.} \qquad \frac{dx}{dr} \quad \leftarrow \text{dep}$$

independent x ind.   ind.

Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

∂ - partial symbol

$\frac{\partial y}{\partial t}$ "partial of y with respect to t"

[1] These are the subject of this course.

Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$

1st order

$$y''' + (y')^4 = x^3$$

3rd order

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

2nd order

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:}\quad \frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \dots\quad \frac{d^ny}{dx^n},\quad \text{or}$$

$$\text{Prime \& superscripts:}\quad y',\quad y'',\quad \dots\quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s},\quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

where F is some real valued function of $n+2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

A couple of normal forms

If $n = 1$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y)$$

Here's an example $\quad \dfrac{dy}{dx} = x^2 \cos(y)$

If $n = 2$, an equation in normal form would look like

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

Here's an example $\quad \dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - xy^2$

Special form for a first order equation

Differential Form: A first order equation may appear in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Either x or y could be considered the independent variable!

We can write this in two normal forms.

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$N(x,y)\,dy = -M(x,y)\,dx$$

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} \quad \text{if } N(x,y) \neq 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$M(x,y)\,dx = -N(x,y)\,dy$$

$$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)}$$

if $M(x,y) \neq 0$

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

- $y, y', \ldots, y^{(n)}$ have to appear as themselves (to the power 1)

An equation that is not linear is <u>Nonlinear</u>

Examples (Linear -vs- Nonlinear)

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

(a) $y''+4y=0$

This is linear with $g(x)=0$

$a_2(x)=1$, $a_1(x)=0$, $a_0(x)=4$

(b) $t^2\dfrac{d^2x}{dt^2}+2t\dfrac{dx}{dt}-x=e^t$

This is linear with $g(t)=e^t$

$a_2(t)=t^2$, $a_1(t)=2t$, $a_0(t)=-1$

Examples (Linear -vs- Nonlinear)

(a) $$\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^4 = x^3$$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^3 \frac{dy}{dx} = x^3$$

↑ this is a nonlinear term

the equation is nonlinear.

Note: $u''+u' = \cos(x)$ would be linear

(b) $$u''+u' = \cos u$$

↑ nonlinear term

the eqn is nonlinear

Exercises

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

$$\text{(a)} \quad y'' + 2ty' = \cos t + y$$

$$y'' + 2ty' - y = \cos t$$

Ind. Var.: t

Dep. Var.: y

Order: 2nd

Linear/Non: Linear

(b) $$\frac{d^3y}{dx^3} + 2y\frac{dy}{dx} = \frac{d^2y}{dx^2} + \tan(x)$$

$$y''' - y'' + \underbrace{2y\,y'}_{\text{nonlinear term}} = \tan x$$

coef. of y' depends on y

Ind. Var.: x

Dep. Var.: y

Order: 3rd

Linear/Non: Nonlinear

(c) $\ddot{\theta}+\frac{g}{\ell}\sin\theta = 0$ (g and ℓ are constant)

θ is dependent so $\sin\theta$ is a nonlinear term

Ind. Var.: time (dot notation) t

Dep. Var.: θ

Order: 2^{nd}

Linear/Non: Non liner

Solution of $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval[2] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

An identity is an equation that is always true

e.g. $0=0$.

For example $\phi(x) = \cos(2x)$ is a soln. of

$y''+4y=0$ on $(-\infty,\infty)$.

we can call this an explicit solution.

[2] The interval is called the *domain of the solution* or the *interval of definition*.

Implicit Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

Recall that implicit differentiation can be used to find $\frac{dy}{dx}$ from the eqn $G(x,y) = 0$.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty)\,, \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

Note $\phi(t) = 3e^{2t}$ is infinitely differentiable.

So ϕ has 2 continuous derivatives on I.

We need to show that the eqn. reduces to an identity on substitution.

Set $y = 3e^{2t}$. Then $\frac{dy}{dt} = 3e^{2t} \cdot 2 = 6e^{2t}$

and $\frac{d^2y}{dt^2} = 6e^{2t} \cdot 2 = 12e^{2t}$

Our eqn is $y'' - y' - 2y = 0$

$$y'' - y' - 2y \stackrel{?}{=} 0$$

$$12e^{2t} - 6e^{2t} - 2(3e^{2t}) \stackrel{?}{=} 0$$

$$12e^{2t} - 6e^{2t} - 6e^{2t} \stackrel{?}{=} 0$$

$$0 = 0$$ an identity

Hence $\phi(t) = 3e^{2t}$ solves the ODE on I.

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(x) = 5\tan(5x), \quad I = \left(-\frac{\pi}{10}, \frac{\pi}{10}\right), \quad y' - 25 = y^2$$

Note: if $-\pi/10 < x < \pi/10$, then $5(-\pi/10) < 5x < 5(\pi/10)$

$\Rightarrow -\pi/2 < 5x < \pi/2$. $\tan(5x)$ is continuous and continuously differentiable on I.

Set $y = 5\tan(5x)$. Then $y' = 5\sec^2(5x)\cdot 5$

$= 25\sec^2(5x)$

$$y' - 25 = y^2$$

$$y' - 25 = y^2$$

$$25\sec^2(5x) - 25 \stackrel{?}{=} \left(5\tan(5x)\right)^2$$

$$25\left(\sec^2(5x) - 1\right) \stackrel{?}{=} 25\tan^2(5x)$$

$$25\tan^2(5x) = 25\tan^2(5x) \quad \text{on } I$$

identity

So $\phi(x) = 5\tan(5x)$ solves the ODE on I.

* Recall $\tan^2\theta + 1 = \sec^2\theta$ ie $\sec^2\theta - 1 = \tan^2\theta$

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(x) = \sqrt{\ln x + 1}, \quad I = (1, \infty), \quad dx - 2xy\,dy = 0$$

We can write the DE in normal form

$2xy\,dy = dx \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{2xy}$

Note for $x > 1$, $\ln x > 0$ and $\ln x$ is continuous.
so $\ln x + 1 > 1$ and $\sqrt{\ln x + 1}$ is defined,
continuous and differentiable.

Set $y = \sqrt{\ln(x) + 1} = (\ln x + 1)^{1/2}$

Then $\frac{dy}{dx} = \frac{1}{2}(\ln x + 1)^{-1/2} \cdot \left(\frac{1}{x} + 0\right)$

so $\frac{dy}{dx} = \frac{1}{2\sqrt{\ln x + 1}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x + 1}}$

$$\frac{dy}{dx} \stackrel{?}{=} \frac{1}{2xy}$$

$$\frac{1}{2x\sqrt{\ln x + 1}} = \frac{1}{2x\sqrt{\ln x + 1}}$$ an identity.

Hence $\phi(x) = \sqrt{\ln x + 1}$ solves the ODE on the interval.

Implicit Solution Example

Verify that the relation defines and implicit solution of the differential equation.

$$y^2 - 2x^2y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

Here, we'll use implicit differentiation to show that the relation being true implies that the ODE is also true.

Do implicit differentiation

$$y^2 - 2x^2y = 1$$

$$2y\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} - 2x^2\cdot\frac{dy}{dx} = 4xy$$

$$2(y-x^2)\frac{dy}{dx} = 4xy \quad \Rightarrow \quad (y-x^2)\frac{dy}{dx} = 2xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy}{y-x^2} \quad (\text{assuming } y-x^2 \neq 0)$$

This is the ODE. Hence $y^2 - 2x^2y = 1$ defines an implicit solution.

Function vs Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take I to be the largest, or one of the largest, possible interval. It may depend on other information.

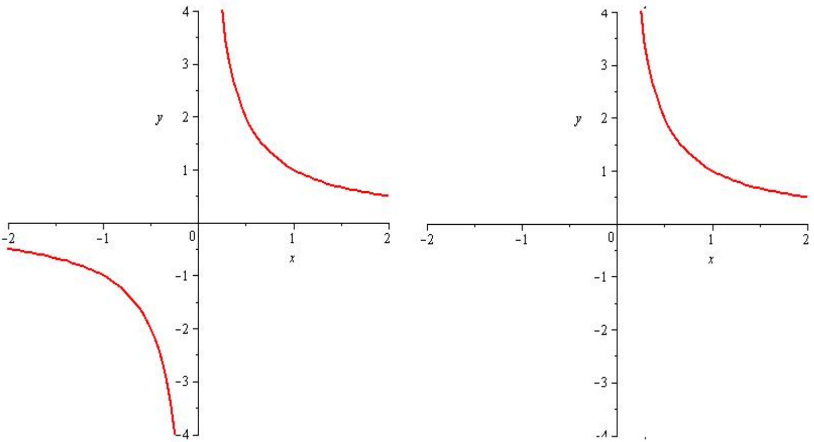

Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

$y = c_1 x + \frac{c_2}{x} = c_1 x + c_2 x^{-1}$

we substitute this in assuming $x \neq 0$

$y' = c_1 + c_2(-x^{-2}) = c_1 - \frac{c_2}{x^2}$

$y'' = 0 + c_2(2x^{-3}) = \frac{2c_2}{x^3}$

$$x^2 y'' + xy' - y \stackrel{?}{=} 0$$

$$x^2\left(\frac{2c_2}{x^3}\right) + x\left(c_1 - \frac{c_2}{x^2}\right) - \left(c_1 x + \frac{c_2}{x}\right) \stackrel{?}{=} 0$$

$$\frac{2c_2}{x} + c_1 x - \frac{c_2}{x} - c_1 x - \frac{c_2}{x} \stackrel{?}{=} 0$$

$$\frac{2c_2}{x} - \frac{c_2}{x} - \frac{c_2}{x} + c_1 x - c_1 x \stackrel{?}{=} 0$$

$$0 + 0 = 0 \quad \text{an identity.}$$

Hence $y = c_1 x + \frac{c_2}{x}$ is a solution for any values of c_1 and c_2.

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $c_1 x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).

Section 2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation [3]

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

the x_0 is the same through out.

[3] on some interval I containing x_0.

First and Second Order Cases

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

1st order

one condition

The curve would pass through (x_0, y_0)

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

If y is position, y'' would be acceleration. y_0 would be initial position and y_1 would be initial velocity.

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2y'' + xy' - y = 0$, solve the IVP

$$x^2y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

We know $y = c_1 x + \frac{c_2}{x}$ solves the ODE. Now we'll insist that $y(1) = 1$ and $y'(1) = 3$.

Set $y(1) = 1$ $\quad y(1) = c_1 \cdot 1 + \frac{c_2}{1} = 1$

Set $y'(1) = 3$ $\quad y'(1) = c_1 - \frac{c_2}{1^2} = 3$ * from slide 28

Solve the system $\quad c_1 + c_2 = 1$

$c_1 - c_2 = 3$

$$\text{add} \quad \begin{array}{r} c_1 + c_2 = 1 \\ c_1 - c_2 = 3 \\ \hline 2c_1 = 4 \end{array} \quad c_1 = 2 \quad \Bigg\} \Rightarrow \quad c_2 = 1 - c_1 = 1 - 2 = -1$$

$$\boxed{\text{The solution to the IVP is} \quad y = 2x - \frac{1}{x}}$$

Example

Part 1

Show that for any nonnegative constant c the relation $x^2+y^2=c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

Using implicit diff. $x^2+y^2=c$ $\quad 2x+2y\frac{dy}{dx}=0$

$2y\frac{dy}{dx}=-2x \implies y\frac{dy}{dx}=-x$

$\implies \frac{dy}{dx}=\frac{-x}{y}$ our ODE

Hence $x^2+y^2=c$ is an implicit solution.

Example

Part 2

Use the preceding results to find an explicit solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

We know that $x^2+y^2=C$ defines a solution to $\frac{dy}{dx} = \frac{-x}{y}$ implicitly. From $y(0)=-2$

$$0^2+(-2)^2 = C \implies C=4$$

So we have $x^2+y^2=4$

Let's find an explicit solution.

$$x^2+y^2=4 \implies y^2=4-x^2$$

so $\quad y=\sqrt{4-x^2} \quad$ or $\quad y=-\sqrt{4-x^2}$

Since $y(0)=-2$ only the one on the right can solve the IVP.

Hence an explicit soln. is

$$y=-\sqrt{4-x^2}$$