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Section 6: Linear Equations Theory and Terminology

We are considering an n^{th} order, linear, homogeneous ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assuming that each a_i is continuous and a_n is never zero on the interval of interest.

Principle of Superposition: Linear, Homogeneous ODE

Theorem: If $y_1, y_2, \dots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \dots, c_k$.

This is called the **principle of superposition.**

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

We note that if the only way to satisfy the above equation is to set each c_i to zero, then the functions are **linearly independent.** If at least one of the c_i can be nonzero, they are **linearly dependent.**

Definition of Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for* each x in I.

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*For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$

We'll use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= e^x(-2e^{-2x}) - e^x(e^{-2x}) = -2e^{-x} - e^{-x} = -3e^{-x}$$

$$W(y_1, y_2)(x) = -3e^{-x} \neq 0$$

So y_1 and y_2 are <u>linearly independent</u>,

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ form a fundamental solution set of the ODE

$$y'' - y = 0 \quad \text{on} \quad (-\infty, \infty),$$

and determine the general solution.

To verify y_1, y_2 form a fund. solution set check

(i) they're solutions

(ii) there are "n" of them ← 2nd order, 2 solns.

(iii) they're lin. independent,

(i) $y_1 = e^x$, $y_1' = e^x$, $y_1'' = e^x$

$$y_1'' - y_1 \stackrel{?}{=} 0$$

$$e^x - e^x \stackrel{?}{=} 0$$

$$0 = 0 \quad \checkmark$$

yes y_1 solves it.

$y_2 = e^{-x}$, $y_2' = -e^{-x}$, $y_2'' = e^{-x}$

$$y_2'' - y_2 \overset{?}{=} 0$$
$$e^{-x} - e^{-x} \overset{?}{=} 0$$
$$0 = 0 \ \checkmark$$

y_2 solves it too

(iii) Let's use the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}$$

$$= e^{x}(-e^{-x}) - e^{x}(e^{-x}) = -1 - 1 = -2$$

Since $W(y_1, y_2)(x) \neq 0$ they are linearly independent.

We conclude that $y_1 = e^x$, $y_2 = e^{-x}$ forms a fundamental solution set.

The general solution to the ODE is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^x + c_2 e^{-x}$$

Consider $x^2y'' - 4xy' + 6y = 0$ for $x > 0$

Determine which if any of the following sets of functions is a fundamental solution set.

(a) $y_1 = 2x^2, \quad y_2 = x^2$ $\leftarrow$ lin. dependent note $1y_1 + (-2)y_2 = 0$ for all $x > 0$

(b) $y_1 = x^{-2}, \quad y_2 = x^2$

(c) $y_1 = x^3, \quad y_2 = x^2$

$\Big\}$ can't eliminate these, so we'll check

(d) $y_1 = x^2, \quad y_2 = x^3, \quad y_3 = x^{-2}$ $\leftarrow$ too many functions since $n=2$

(b) Check $y_1 = x^{-2}$, $y_1' = -2x^{-3}$, $y_1'' = 6x^{-4}$

$$x^2y_1'' - 4xy_1' + 6y_1 \stackrel{?}{=} 0$$

$$x^2(6x^{-4}) - 4x(-2x^{-3}) + 6x^{-2} \stackrel{?}{=} 0$$

$$6x^{-2} + 8x^{-2} + 6x^{-2} \stackrel{?}{=} 0$$

$$20x^{-2} \neq 0$$

y_1 doesn't solve the ODE

(b) is not the correct option.

Check (c) $\quad x^2y'' - 4xy' + 6y = 0$

$$y_1 = x^3, \quad y_1' = 3x^2, \quad y_1'' = 6x$$

$$x^2y_1'' - 4xy_1' + 6y_1 \stackrel{?}{=} 0$$

$$x^2(6x) - 4x(3x^2) + 6x^3 \stackrel{?}{=} 0$$

$$6x^3 - 12x^3 + 6x^3 \stackrel{?}{=} 0$$

$$0 = 0$$

yes, y_1 solves the ODE

$$y_2 = x^2, \quad y_2' = 2x, \quad y_2'' = 2$$

$$x^2 y_2'' - 4x y_2' + 6y_2 \stackrel{?}{=} 0$$

$$x^2(2) - 4x(2x) + 6x^2 \stackrel{?}{=} 0$$

$$2x^2 - 8x^2 + 6x^2 \stackrel{?}{=} 0$$

$$0 = 0$$

yes y_2 solves the ODE too.

Lastly, we check for linear dependence.

Use the Wronskian

$$W(y_1,y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix}$$

$$= x^3(2x) - 3x^2(x^2) = 2x^4 - 3x^4 = -x^4$$

Since $W(y_1,y_2)(x) \neq 0$, they are lin. independent,

Option (c) is a fundamental solution set.

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Write the associated homogeneous equation

(a) $x^3y''' - 2x^2y'' + 3xy' + 17y = e^{2x}$

$$x^3y''' - 2x^2y'' + 3xy' + 17y = 0$$

(b) $\dfrac{d^2y}{dx^2} + 14\dfrac{dy}{dx} = \cos\left(\dfrac{\pi x}{2}\right)$

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} = 0$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

where $y_c = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$.

Another Superposition Principle (for nonhomogeneous eqns.)

Let y_{p_1}, y_{p_2}, ..., y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g_i(x)$$

for $i=1,\ldots,k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p=y_{p_1}+y_{p_2}+\cdots+y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n}+\cdots+a_0(x)y=g_1(x)+g_2(x)+\cdots+g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(a) Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1} = 6, \quad y_{p_1}' = 0 \quad y_{p_1}'' = 0$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6y_{p_1} \stackrel{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$$

$$36 = 36 \quad \checkmark \text{ true}$$

so y_{p_1} does solve $x^2y'' - 4xy' + 6y = 36$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2} = -7x$, $y_{p_2}' = -7$, $y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x\, y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$$

$$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$$

$$28x - 42x \stackrel{?}{=} -14x$$

$$-14x = -14x \quad \text{true}$$

Yes $y_{p_2} = -7x$ solves $x^2y'' - 4xy' + 6y = -14x$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) Recall that $y_1 = x^3$ and $y_2 = x^2$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$. (Verify that your result is correct.)

By the principle of superposition for nonhomogeneous equations

$$y_p = y_{p_1} + y_{p_2}$$

$$y_p = 6 - 7x$$

Also, $y_c = c_1 y_1 + c_2 y_2 = c_1 x^3 + c_2 x^2$

The general solution to $x^2y'' - 4xy' + 6y = 36 - 14x$ is

$$y = c_1 x^3 + c_2 x^2 + 6 - 7x$$

$* \quad y = y_c + y_p$

Let's verify this is correct.

$$y' = 3c_1 x^2 + 2c_2 x + 0 - 7 \quad = 3c_1 x^2 + 2c_2 x - 7$$

$$y'' = 6c_1 x + 2c_2 - 0 = 6c_1 x + 2c_2$$

$$x^2 y'' - 4x y' + 6y \overset{?}{=} 36 - 14x$$

$$x^2(6c_1x+2c_2)-4x(3c_1x^2+2c_2x-7)+6(c_1x^3+c_2x^2+6-7x) \stackrel{?}{=} 36-14x$$

$$6c_1x^3+2c_2x^2-12c_1x^3-8c_2x^2+28x+6c_1x^3+6c_2x^2+36-42x \stackrel{?}{=} 36-14x$$

$$c_1x^3(6-12+6)+c_2x^2(2-8+6)+28x-42x+36 \stackrel{?}{=} 36-14x$$

$$0+0-14x+36 \stackrel{?}{=} 36-14x$$

$$-14x+36 = 36-14x \quad \checkmark$$

Yes, our solution is correct!

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

From before $\quad y = c_1x^3 + c_2x^2 + 6 - 7x$

$y' = 3c_1x^2 + 2c_2x - 7$

$y(1) = c_1(1)^3 + c_2(1)^2 + 6 - 7(1) = 0 \quad \Rightarrow \quad c_1 + c_2 = 1$

$y'(1) = 3c_1(1)^2 + 2c_2(1) - 7 = -5 \quad \Rightarrow \quad 3c_1 + 2c_2 = 2$

$$3c_1 + 3c_2 = 3$$
$$-\,(\,3c_1 + 2c_2 = 2\,)$$
$$c_2 = 1$$

$$c_1 = 1 - c_2 = 1 - 1 = 0$$

$$\boxed{\text{The solution to the IVP is} \quad y = x^2 + 6 - 7x}$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x) y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

See slides 10, 11 for the verification part.

Set $y_2 = e^{-x}u(x)$. Then

$$y_2' = -e^{-x}u(x) + e^{-x}u'(x)$$

$$y_2'' = e^{-x}u(x) - e^{-x}u'(x) - e^{-x}u'(x) + e^{-x}u''(x)$$

$$= e^{-x}u(x) - 2e^{-x}u'(x) + e^{-x}u''(x)$$

We require $y_2'' - y_2 = 0$

$$y_2'' - y_2 = e^{-x}\cancel{u(x)} - 2e^{-x}u'(x) + e^{-x}u''(x) - e^{-x}\cancel{u(x)} = 0$$

$$e^{-x}\left(-2u'(x) + u''(x)\right) = 0$$

$$\Rightarrow \quad u''(x) - 2u'(x) = 0$$

If we set $w = u'(x)$, then we get a 1st order separable eqn.

$$w' - 2w = 0$$

Solve this $\qquad \dfrac{dw}{dx} = 2w \implies \dfrac{1}{w}\dfrac{dw}{dx} = 2$

$\displaystyle\int \frac{1}{w}\,dw = \int 2\,dx$ assuming $w > 0$

$\ln w = 2x + C \implies w = Ae^{2x}$ where $A = e^{C}$ *

Let's let $A = 2$ so $w = 2e^{2x}$. As $w = u'$

$$u = \int w\,dx = \int 2e^{2x}\,dx = e^{2x}$$

* A is any nonzero constant.

Recall that $y_2 = uy_1 = e^{-x}u(x)$. So

$$y_2 = e^{-x}\left(e^{2x}\right) = e^{x}$$

So our solutions are $y_1 = e^{-x}$, $y_2 = e^{x}$.

The general solution to $y'' - y = 0$ is

$$\boxed{y = C_1 e^{-x} + C_2 e^{x}}$$

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

Suppose $y_2 = y_1 u = y_1(x)u(x)$

then $y_2' = y_1' u + y_1 u'$

$y_2'' = y_1'' u + y_1' u' + y_1' u' + y_1 u''$

$= y_1'' u + 2y_1' u' + y_1 u''$

Plug into the ODE

$$y_2'' + P(x) y_2' + Q(x) y_2 = 0$$

$$\underline{y_1'' u} + 2 \underline{y_1' u'} + \underline{y_1 u''} + P(x)\left(\underline{y_1' u} + \underline{y_1 u'}\right) + \underline{Q(x) y_1} u = 0$$

$$\underline{y_1 u''} + \underline{(2y_1' + P(x) y_1)} u' + \underline{(y_1'' + P(x) y_1' + Q(x) y_1)} u = 0$$

* y_1 solves $y_1'' + P(x) y_1' + Q(x) y_1 = 0$ *

$$y_1 u'' + (2y_1' + P(x) y_1) u' = 0$$

Let $w = u'$. Divide by y_1 (assume its ok)

Then w solves the 1^{st} order eqn (and separable)

$$w' + \left(2\frac{y_1'}{y_1} + P(x)\right)w = 0$$

Separate and Solve

$$\frac{dw}{dx} = -\left(2\frac{y_1'}{y_1} + P(x)\right)w$$

$$\frac{1}{w}\,dw = -\left(2\frac{y_1'}{y_1} + P(x)\right)dx$$

$$= -2\frac{dy_1}{y_1} - P(x)\,dx$$

$*\ \frac{dy_1}{dx}dx = dy_1$

$$\int \frac{1}{w}\,dw = -2\int \frac{dy_1}{y_1} - \int p(x)\,dx$$

$$\ln w = -2\ln|y_1| - \int p(x)\,dx$$

$$\ln w = \ln y_1^{-2} - \int p(x)\,dx$$

$$e^{\ln w} = e^{\left(\ln y_1^{-2} - \int p(x)dx\right)}$$

$$w = y_1^{-2}\, e^{-\int p(x)\,dx} = \frac{e^{-\int p(x)\,dx}}{y_1^2}$$

Since $w = u'$,

$$u = \int w\,dx = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Finally,

$$y_2 = y_1 u = y_1 \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2 \qquad x > 0$$

$y_2 = y_1 u$ where $u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\, dx$

Standard form $\quad y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0$

$P(x) = \frac{-3}{x}, \quad -\int P(x)dx = -\int \frac{-3}{x} dx = 3\ln x = \ln x^3$

So $\quad e^{-\int P(x)dx} = e^{\ln x^3} = x^3$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx = \int \frac{x^3}{(x^2)^2}\,dx = \int \frac{x^3}{x^4}\,dx$$

$$= \int \frac{1}{x}\,dx = \ln x$$

So $\quad y_2 = y_1 u = x^2 \ln x$

$$\boxed{\text{The general solution to the ODE is} \quad y = c_1 x^2 + c_2 x^2 \ln x.}$$

Let's verify that ① y_2 solves the ODE and ② that y_1, y_2 are lin. independent,

$$y_2 = x^2 \ln x$$

$$y_2' = 2x \ln x + x^2\left(\frac{1}{x}\right) = 2x\ln x + x$$

$$y_2'' = 2\ln x + 2x \cdot \frac{1}{x} + 1 = 2\ln x + 3$$

$$x^2 y_2'' - 3x y_2' + 4y_2 \stackrel{?}{=} 0$$

$$x^2(2\ln x + 3) - 3x(2x\ln x + x) + 4x^2 \ln x \stackrel{?}{=} 0$$

$$2x^2\ln x + 3x^2 - 6x^2\ln x - 3x^2 + 4x^2\ln x \stackrel{?}{=} 0$$

$$x^2\ln x\,(2-6+4)+x^2(3-3)\stackrel{?}{=}0$$

$$0=0$$

y_2 does solve the ODE

Using the Wronskian

$$W(y_1,y_2)(x)=\begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=\begin{vmatrix} x^2 & x^2\ln x\\ 2x & 2x\ln x+x\end{vmatrix}$$

$$=x^2(2x\ln x+x)-2x\,(x^2\ln x)$$

$$=2x^3\ln x+x^3-2x^3\ln x\qquad = x^3$$

$$W(y_1, y_2)(x) = x^3 \neq 0$$

So y_1, y_2 are linearly independent.

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 3, \quad y'(0) = -2$$

$y_2 = y_1 u$ where $u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$. $P(x) = 4$

$$e^{-\int P(x)\,dx} = e^{-\int 4\,dx} = e^{-4x}$$

$$u = \int \frac{e^{-4x}}{(e^{-2x})^2}\,dx = \int \frac{e^{-4x}}{e^{-4x}}\,dx = \int dx = x$$

So $y_2 = xe^{-2x}$ and the general solution is

$$y = c_1 e^{-2x} + c_2 x e^{-2x}.$$

Apply $y(0) = 3$, $y'(0) = -2$. $\quad y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$

$$y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = 3$$

$$c_1 \qquad = 3$$

$$y'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 \cdot 0 e^0 = -2$$

$$-2 \cdot 3 + c_2 = -2 \quad \Rightarrow \quad c_2 = -2 + 6 = 4$$

The soln. to the IVP is

$$y = 3e^{-2x} + 4xe^{-2x}$$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Question: What sort of function y could be expected to satisfy

$$y'' = \text{constant}\,y' + \text{constant}\,y?$$

We look for solutions of the form $y = e^{mx}$ with m constant.

$$ay'' + by' + cy = 0$$

$$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

We require $\quad ay'' + by' + cy = 0$

$$a(m^2 e^{mx}) + b(me^{mx}) + ce^{mx} = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

This will be true if m satisfies

$$am^2 + bm + c = 0$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$