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Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with **constant coefficients**

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

We assumed that $y = e^{mx}$ for constant m and found that such a function does solve the ODE provided m is a root of the quadratic equation

$$am^2 + bm + c = 0$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

We'll use the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= e^{m_1 x}\left(m_2 e^{m_2 x}\right) - m_1 e^{m_1 x}\left(e^{m_2 x}\right)$$

$$= e^{(m_1+m_2)x}\left(m_2 - m_1\right) \neq 0$$

since $m_2 \neq m_1$

Since $W(y_1, y_2)(x) \neq 0$, y_1 and y_2 are linearly independent.

Example

Find the general solution of the ODE

$$y'' - 2y' - 2y = 0$$

Characteristic eqn. $m^2 - 2m - 2 = 0$

Let's complete the square

$$m^2 - 2m + 1 - 1 - 2 = 0$$

$$(m-1)^2 - 3 = 0 \Rightarrow (m-1)^2 = 3$$

$$m - 1 = \pm\sqrt{3}$$

$$m = 1 \pm \sqrt{3}$$

2 distinct real roots

$m_1 = 1+\sqrt{3}$, $m_2 = 1-\sqrt{3}$

$y_1 = e^{(1+\sqrt{3})x}$ and $y_2 = e^{(1-\sqrt{3})x}$

The general solution is

$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$$

Example

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

Char. Eqn $\quad m^2 + m - 12 = 0$

$(m+4)(m-3) = 0 \implies m = -4$ or $m = 3$

2 distinct real roots

$m_1 = -4, \quad m_2 = 3$

$y_1 = e^{-4x}$ and $y_2 = e^{3x}$

general soln. $\quad y = C_1 e^{-4x} + C_2 e^{3x}$

$y = C_1 e^{-4x} + C_2 e^{3x}$ $\quad y(0) = 1 \ , \ y'(0) = 10$

$y' = -4C_1 e^{-4x} + 3C_2 e^{3x}$

$y(0) = C_1 e^0 + C_2 e^0 = 1 \implies C_1 + C_2 = 1$

$y'(0) = -4C_1 e^0 + 3C_2 e^0 = 10 \implies -4C_1 + 3C_2 = 10$

$$\begin{array}{l} 4C_1 + 4C_2 = 4 \\ -4C_1 + 3C_2 = 10 \\ \text{add} \quad \overline{ 7C_2 = 14} \implies C_2 = 2 \end{array}$$

$C_1 = 1 - C_2 = 1 - 2 = -1$

The soln. to the IVP is

$$y = -e^{-4x} + 2e^{3x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

Standard form $\quad y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad P(x) = \frac{b}{a}$

$y_2 = y_1 u \qquad u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

$$u = \int \frac{e^{-\int \frac{b}{a}dx}}{\left(e^{-\frac{bx}{2a}}\right)^2}\,dx = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx$$

$$= \int dx = x$$

$$y_2 = y_1 u = e^{-\frac{bx}{2a}} \cdot x = x\,e^{-\frac{b}{2a}x}$$

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Char. Eqn $\quad 4m^2 - 4m + 1 = 0$

$(2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}$ repeated root.

$y_1 = e^{\frac{1}{2}x}$ and $y_2 = x e^{\frac{1}{2}x}$

The general solution is $y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}$

Example

Solve the IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

Char. Eqn $\quad m^2 + 6m + 9 = 0 \Rightarrow (m+3)^2 = 0$

$m = -3$ repeated root.

$y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$

General solution: $\quad y = c_1 e^{-3x} + c_2 x e^{-3x}$

$$y'(x) = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 e^0 = 4 \implies c_1 = 4$$

$$y'(0) = -3c_1 e^0 + c_2 e^0 - 3c_2 \cdot 0 e^0 = 0$$

$$-3 \cdot 4 + c_2 = 0 \implies c_2 = 12$$

The solution to the IVP is

$$\boxed{y = 4e^{-3x} + 12xe^{-3x}}$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$Y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$Y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

Using the principle of superposition

Let $y_1 = \frac{1}{2}Y_1 + \frac{1}{2}Y_2 = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$

Let $y_2 = \frac{1}{2i}Y_1 - \frac{1}{2i}Y_2 = \frac{1}{2i}\left(2i\, e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$

So $y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$

and the general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Char. Eqn $\quad m^2 + 4m + 6 = 0$

Quadratic formula $\quad m = \dfrac{-4 \pm \sqrt{4^2 - 4\cdot 1\cdot 6}}{2}$

$= \dfrac{-4 \pm \sqrt{-8}}{2} = \dfrac{-4 \pm 2\sqrt{2}\, i}{2}$

$= -2 \pm \sqrt{2}\, i$

Note $\alpha \pm i\beta = -2 \pm \sqrt{2}\, i$

$\Rightarrow \alpha = -2$ and $\beta = \sqrt{2}$

$i = \sqrt{-1}$

$$y_1 = e^{-2x}\cos(\sqrt{2}\,x) \qquad y_2 = e^{-2x}\sin(\sqrt{2}\,x)$$

The general solution is

$$y = c_1 e^{-2x}\cos(\sqrt{2}\,x) + c_2 e^{-2x}\sin(\sqrt{2}\,x)$$

Example

Solve the IVP

$$y'' + 4y = 0, \quad y(0) = 3, \quad y'(0) = -5$$

Char. Eqn $\quad m^2 + 4 = 0 \quad \Rightarrow \quad m^2 = -4$

$m = \pm\sqrt{-4} = \pm\sqrt{4}\, i = \pm 2i$

$\alpha \pm i\beta = \pm 2i$

$\alpha = 0 \qquad \beta = 2$

$y_1 = e^{0x}\cos(2x) = \cos(2x)$

$y_2 = e^{0x}\sin(2x) = \sin(2x)$

general solution $\quad y = c_1 \cos(2x) + c_2 \sin(2x)$

Apply $y(0) = 3$, $y'(0) = -5$

$$y'(x) = -2c_1 \sin(2x) + 2c_2 \cos(2x)$$

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = 3 \implies c_1 = 3$$

$$y'(0) = -2c_1 \sin 0 + 2c_2 \cos 0 = -5 \implies c_2 = \frac{-5}{2}$$

The soln to the IVP is

$$\boxed{y = 3\cos(2x) - \frac{5}{2}\sin(2x)}$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
- If a root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Claim: The char. eqn should be

$m^3 - 4m = 0$

Solve the ODE

$$y''' - 4y' = 0$$

Let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$, $y''' = m^3 e^{mx}$

So

$$y''' - 4y' = m^3 e^{mx} - 4me^{mx} = 0$$

$$e^{mx}(m^3 - 4m) = 0$$

$$\Rightarrow \quad m^3 - 4m = 0$$

Our Characteristic Eqn is

$$m^3 - 4m = 0$$

$$m(m^2 - 4) = 0$$

$$m(m-2)(m+2) = 0$$

3 distinct real roots $m_1 = 0$, $m_2 = 2$, $m_3 = -2$

Three solutions to this 3^{rd} order eqn are

$y_1 = e^{m_1 x} = e^{0x} = 1$, $y_2 = e^{m_2 x} = e^{2x}$, and $y_3 = e^{m_3 x} = e^{-2x}$

The general solution is

$$y = C_1 + C_2 e^{2x} + C_3 e^{-2x}$$

Example

3rd order eqn. needs 3 lin. independent solutions.

Solve the ODE

$$y''' - 3y'' + 3y' - y = 0$$

Char. Eqn $m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)^3 = 0$

$m = 1$ repeated triple root!

$y_1 = e^{mx} = e^{x}$

$y_2 = xe^{mx} = xe^{x}$ and $y_3 = x^2e^{mx} = x^2e^{x}$

The general solution is

$$y = c_1 e^{x} + c_2 x e^{x} + c_3 x^2 e^{x}$$

Section 9: Method of Undetermined Coefficients

The context here is **linear, constant coefficient**, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We can consider what kind of function y_p would produce the 1^{st} degree polynomial $g(x) = 8x+1$ when plugged into $y''-4y'+4y$. Let's guess that y_p is a 1^{st} degree polynomial

$$y_p = Ax + B \quad , \; A, B \text{ - constants}$$

$$y'' - 4y' + 4y = 8x + 1$$

Guess $y_p = Ax + B$ sub this into the ODE

$y_p' = A$, $y_p'' = 0$

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$0 - 4A + 4(Ax + B) = 8x + 1$$

$$4Ax + (-4A + 4B) = 8x + 1$$

This is true if $4A = 8$ and $-4A + 4B = 1$

$4A = 8 \Rightarrow A = 2$ so $-4A + 4B = 1 \Rightarrow 4B = 1 + 4A$

$= 1 + 8 = 9$

$B = \frac{9}{4}$

$\boxed{y_p = 2x + \frac{9}{4} \text{ is a particular solution}}$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{3x}$$

The "form" of g is "constant times e^{3x}."

Let's Guess that $y_p = Ae^{3x}$

Substitute into the left : $y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$

$$y_p'' - 4y_p' + 4y_p = 6e^{3x}$$

$$9Ae^{3x} - 4(3Ae^{3x}) + 4(Ae^{3x}) = 6e^{3x}$$

$$9Ae^{3x} - 12Ae^{3x} + 4Ae^{3x} = 6e^{3x}$$

$$Ae^{3x} = 6e^{3x}$$

This holds if $A = 6$.

So the particular solution is $y_p = 6e^{3x}$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

If the "form" of $g(x) = 16x^2$ is "a constant times x^2"

we might guess that

$$y_p = Ax^2$$

substitute : $y_p' = 2Ax$, $y_p'' = 2A$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

Matching coefficients

$$\left.\begin{array}{r} 4A = 16 \\ -8A = 0 \\ 2A = 0 \end{array}\right\}$$ not solvable, requires $A = 4$ AND $A = 0$

The guess $y_p = Ax^2$ fails. The correction is to recognize $g(x) = 16x^2$ as a 2^{nd} degree polynomial.

Let's guess $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B, \qquad y_p'' = 2A$$

Substitute

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Match Coefficients

$$4A = 16$$

$$-8A + 4B = 0$$

$$2A - 4B + 4C = 0$$

$A = 4, \quad 4B = 8A \Rightarrow B = 2A = 8$

$4C = 4B - 2A \Rightarrow C = B - \frac{1}{2}A = 8 - 2 = 6$

This works. Our particular solution is

$$\boxed{y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If the form of $g(x) = 20\sin(2x)$ is "constant times $\sin(2x)$"

we'd guess $y_p = A\sin(2x)$

Substitute $y_p' = 2A\cos(2x)$, $y_p'' = -4A\sin(2x)$

$$y_p'' - y_p' = 20\sin(2x)$$

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x)$$

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x) + 0\cdot\cos(2x)$$

Match like terms

$$\left.\begin{aligned} -4A &= 20 \\ -2A &= 0 \end{aligned}\right\} \text{not solvable}$$

The guess $y_p = A\sin(2x)$ doesn't work.

We need to consider $g(x)$ as a sum of sine and cosine of $2x$. The correct guess is

$$y_p = A\sin(2x) + B\cos(2x)$$

$$y_p' = 2A\cos(2x) - 2B\sin(2x)$$

$$y_p'' = -4A\sin(2x) - 4B\cos(2x)$$

$$y_p'' - y_p' = 20\sin(2x)$$

$$-4A\sin(2x) - 4B\cos(2x) - \left(2A\cos(2x) - 2B\sin(2x)\right) = 20\sin(2x)$$

$$(-4A+2B)\sin(2x) + (-2A-4B)\cos(2x) = 20\sin(2x) + 0\cdot\cos(2x)$$

Match coeff.

$$-4A+2B = 20 \quad , \quad -2A-4B = 0$$

times 2

$$-8A+4B = 40$$

$$-10A = 40 \qquad A = -4$$

$$4B = -2A = 8 \Rightarrow B = 2$$

This worked and

$$y_p = -4\sin(2x) + 2\cos(2x)$$

Method of Undetermined Coefficients: Observations

- We start by guessing that y_p has the same **form** as $g(x)$
- **"Form"** is meant in a general sense.
- We account for all *like terms* that can arise in derivatives.
- This only works with *constant coefficient* left hand sides!
- This only works with right hand sides whose derivatives terminate or repeat. (polynomial, exponential, sine/cosine, and their sums or products)

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any nonzero constant)

constant $\quad y_p = A$

(b) $g(x) = x - 7$

1st degree $\quad y_p = Ax + B$

(c) $g(x) = 5x$

1st degree $\quad y_p = Ax + B$

(d) $g(x) = 3x^3 - 5$

$y_p = Ax^3 + Bx^2 + Cx + D$

cubic

More Trial Guesses

(e) $g(x) = xe^{3x}$ $\qquad y_p = (Ax+B)e^{3x}$

1st degree poly $\times\ e^{3x}$

(f) $g(x) = \cos(7x)$ $\qquad y_p = A\cos(7x) + B\sin(7x)$

(g) $g(x) = \sin(2x) - \cos(4x)$

$y_p = A\sin(2x) + B\cos(2x) + C\cos(4x) + D\sin(4x)$

(h) $g(x) = x^2\sin(3x)$

$y_p = (Ax^2+Bx+C)\sin(3x) + (Dx^2+Ex+F)\cos(3x)$