June 27 Math 2306 sec 52 Summer 2016

Section 9: Method of Undetermined Coefficients

The context here is **linear, constant coefficient**, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

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Method of Undetermined Coefficients: Observations

- We start by guessing that y_p has the same form as g(x)
- "Form" is meant in a general sense.
- ► We account for all *like terms* that can arise in derivatives.
- > This only works with *constant coefficient* left hand sides!
- This only works with right hand sides whose derivatives terminate or repeat. (polynomial, exponential, sine/cosine, and their sums or products)

Examples of Forms of y_p based on g (Trial Guesses)

(i)
$$g(x) = e^{x} \cos(2x)$$

form $e^{x} \tan e^{x}$ sine on assue of $2x$
 $\vartheta_{p} = Ae^{x} \cos(2x) + Be^{x} \sin(2x)$
(j) $g(x) = x^{3}e^{8x}$ form cubic polynomial times e^{8x}
 $J_{p} = (Ax^{3} + Bx^{2} + (x + D))e^{8x}$
(k) $g(x) = xe^{-x} \sin(\pi x)$ form 1^{s1} degree poly. times e^{x} times
sine on assue of πx

$$y_{P} = (A_{X}+B)e^{-X}Sin(\pi x) + (C_{X}+D)e^{-X}Cos(\pi x)$$

The Superposition Principle

$$y'' - y' = 20\sin(2x) + 4e^{-5x}$$

Recall that we had considered the equation $y'' - y' = 20 \sin(2x)$. We guessed that $y_p = A \sin(2x) + B \cos(2x)$ and then determined that A = -4 and B = 2.

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We found
$$y_{P_1} = -4 \sin(2x) + 2 \cos(2x)$$

Now find $y_{P_2} \cdot y'' - y' = 4e^{5x}$
Guess $y_{P_1} = Ae^{-5x}$
 $y_{P_2} = -5Ae^{-5x}$, $y_{P_2} = 25Ae^{-5x}$
 $y_{P_2} = -5Ae^{-5x}$, $y_{P_2} = 25Ae^{-5x}$
 $y_{P_2} = -5Ae^{-5x}$, $y_{P_2} = 25Ae^{-5x}$

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This holds if $30A = Y \Rightarrow A = \frac{Y}{30} = \frac{2}{15}$ S. 3r, = 15 e The particular solution to 5"-3' = 20 Sin (2x) + 48 is $y_p = y_{p_1} + y_{p_2}$ $y_{1} = -4 Sin(5x) + 2 cos(5x) + \frac{2}{15}e^{-5x}$

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A Glitch!

$$y'' - y' = 3e^{x}$$

S(x)= 3e^{x} = constant times e^{x}.
Guess $y_{P} = Ae^{x}$
 $y_{P}' = Ae^{x}$, $y_{P}'' = Ae^{x}$
 $y_{P}'' - y_{P}' = Ae^{x} - Ae^{x} = 3e^{x}$
 $OAe^{x} = 3e^{x}$ requires
 $OA = 3e^{x}$

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There's no value of A that satisfier OA=3.
Conside the associated homoseneous eqn.

$$y'' - y' = 0$$

The solutions are $y_i = e^x$ and $y_2 = 1$
so $y_c = c_i e^x + c_2$.
Hen $g(x)$ duplicates part of y_c , the
complementary solution.

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We'll consider cases

Using superposition as needed, begin with the assumption:

 $y_p = y_{p_1} + \cdots + y_{p_k}$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c . Then this first form will suffice.

Case II: y_{ρ} has a term y_{ρ_i} that duplicates a term in the complementary solution y_c . Multiply that term by x^n , where *n* is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

$$y''-2y'+y=-4e^{x}$$

Find be: solve y" - 25' + y = 0 Charc. eqn $m^2 - 2m + 1 = 0 \Rightarrow (m - 1)^2 = 0 \Rightarrow m = 1$, we y,= e, y2= xe, y2= c, e+ c2xe Now find yp: gw=-4e guess yp=Ae e port of wind Try again yp = Axe < part of yc, Still work ▲■▶ ▲ 国▶ ▲ 国▶ - 国 - の々で June 23, 2016

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Try again
$$y_{p} = Ax^{2} e^{x} + 4$$
 this is the correct
form
Substitute into $y'' - 2y' + y = -4e^{x}$
 $y_{p} = Ax^{2} e^{x}$
 $y_{p}' = Ax^{2} e^{x} + 2Ax e^{x}$
 $y_{p}'' = Ax^{2} e^{x} + 2Ax e^{x} + 2Axe^{x} + 2Ae^{x}$
 $z = Ax^{2} e^{x} + 4Axe^{x} + 2Ae^{x}$
 $z = Ax^{2} e^{x} + 4Axe^{x} + 2Ae^{x}$

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$$Ax^{2}e^{+}+YAxe^{+}+2Ae^{-}-z(Ax^{2}e^{+}+2Axe^{+})+Ax^{2}e^{+}=-Ye^{+}e^{-}$$

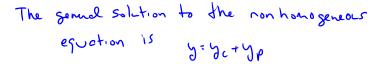
$$x^{2} \stackrel{\times}{e} (A - 2A + A) + x \stackrel{\times}{e} (YA - YA) + 2A \stackrel{\times}{e} = -Y \stackrel{\times}{e}$$

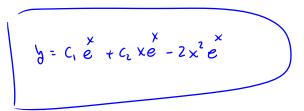
$$\stackrel{\circ}{\sigma} \qquad \stackrel{\circ}{\sigma} \qquad 2A \stackrel{\times}{e} = -Y \stackrel{\times}{e}$$

$$\Rightarrow 2A \stackrel{\times}{e} - Y \Rightarrow A \stackrel{\times}{e} - Z$$

$$s_{0} \quad y_{0} = -Z x^{2} \stackrel{\times}{e}$$

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Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find bc:
$$y'' - 4y' + 4y = 0$$
 $m^2 - 4m + 4z = 0$
 $(m - z)^2 = 0 \Rightarrow m = z root$
 $y_1 = e^{2x} y_2 = xe^{2x}$
 $y_2 = c_1 e^{2x} + c_2 x e^{2x}$

For
$$y_{P_1}$$
: $g_1(x) = Sin(4x)$
 $y_{P_1} = ASin(4x) + BCos(4x) + this correct for Str$

For
$$y_{P_2}$$
: $y_2(x) = xe^{2x}$
 $|^{s1} y_{V_3}(x) = (C_x + D)e^{2x} + Dvplicetus y_{C_3}$
 $y_{P_3}^{n_3} y_{P_2} = (C_x + D)xe^{2x} = (C_x^2 + D_x)e^{2x} + \frac{1}{2} \frac{1}$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find be: y''-y''+y'-y=0 Charc. Eqn $m^3 - m^2 + m - 1 = 0$ factor by grouping m2(m-1) + (m-1) = () $(m^{2}+1)(m-1) = 0$ ⇒ m²=-1, m= ± √-1 = ± ú, q=0 β=1 $\alpha = 1$

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$$\begin{aligned} y_1 &= e^{x} \cos(x) , \quad y_2 &= e^{x} \sin(x) , \quad y_3 &= e^{x} \\ y_2 &= c_1 \cos x + c_2 \sin x + c_3 e^{x} \\ w &= y_1(x) = c_{osx} \quad \text{and} \quad y_2(x) &= x^{4} \\ w &= y_1(x) = c_{osx} \quad \text{and} \quad y_2(x) &= x^{4} \\ For &= y_1(x) , \quad guess \\ &= y_{e_1} &= (A \ C_{orx} + B \ Sinx) x \\ &= A_x \ C_{osx} + B x \ Sinx \quad correct \ form \end{aligned}$$

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For the whole problem

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$$y'' - 2y' + 5y = e^{x} + 7\sin(2x)$$

ind \Im_{c} : $\Im'' - 7\Im' + 5\Im = 0$, $m^{2} - 2m + 5 = 0$
Complete the square
 $m^{2} - 2m + 1 + 4 = 0$
 $(m - 1)^{2} + 4 = 0 \implies (m - 1)^{2} = -4$
 $m - 1 = \pm \sqrt{-4} = \pm 2i$
 $m = 1 \pm 2i$ $d = 1$, $\beta = 2$

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$$y_{1} = e^{x} \cos(2x) \text{ and } y_{2} = e^{x} \sin(2x)$$

$$y_{c} = c_{1}e^{x} \cos(2x) + c_{2}e^{x} \sin(2x)$$

$$F_{c} = S_{1}(x) = e^{x}, \quad \text{suess} \quad y_{p_{1}} = Ae^{x} \in \bigcup_{r \in I} e^{x}$$

$$g_{2}(x) = F \sin(2x)$$

$$g_{r} = B \sin(2x) + C \cos(2x)$$

$$w_{0} - ks \quad ar \quad w_{r} \text{ itten}$$

$$S_{0} = Ae^{x} + B \sin(2x) + C \cos(2x)$$

$$\lim_{r \to 0} e^{x} \exp(2x) = 21/47$$