June 27 Math 2306 sec 52 Summer 2016

Section 9: Method of Undetermined Coefficients

The context here is **linear, constant coefficient**, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Method of Undetermined Coefficients: Observations

- We start by guessing that y_p has the same **form** as $g(x)$
- **"Form"** is meant in a general sense.
- We account for all *like terms* that can arise in derivatives.
- This only works with *constant coefficient* left hand sides!
- This only works with right hand sides whose derivatives terminate or repeat. (polynomial, exponential, sine/cosine, and their sums or products)

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$

form e^x times sine or cosine of $2x$

$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$

(j) $g(x) = x^3 e^{8x}$

form cubic polynomial times e^{8x}

$y_p = (Ax^3 + Bx^2 + Cx + D)\, e^{8x}$

(k) $g(x) = xe^{-x}\sin(\pi x)$

form 1st degree poly. times e^{-x} times sine or cosine of πx

$y_p = (Ax+B)e^{-x}\sin(\pi x) + (Cx+D)e^{-x}\cos(\pi x)$

The Superposition Principle

$$y'' - y' = 20\sin(2x) + 4e^{-5x}$$

Recall that we had considered the equation $y'' - y' = 20\sin(2x)$. We guessed that $y_p = A\sin(2x) + B\cos(2x)$ and then determined that $A = -4$ and $B = 2$.

The principle of superposition says that we can find $y_p = y_{p_1} + y_{p_2}$ where y_{p_1} solves

$y'' - y' = 20\sin(2x)$ and y_{p_2} solves

$y'' - y' = 4e^{-5x}$

We found $y_{p_1} = -4\sin(2x) + 2\cos(2x)$

Now find y_{p_2}. $\quad y'' - y' = 4e^{-5x}$

Guess $y_{p_2} = Ae^{-5x}$

$y_{p_2}' = -5Ae^{-5x}, \quad y_{p_2}'' = 25Ae^{-5x}$

$$y_{p_2}'' - y_{p_2}' = 25Ae^{-5x} - (-5A)e^{-5x} = 4e^{-5x}$$

$$30Ae^{-5x} = 4e^{-5x}$$

This holds if $30A = 4 \Rightarrow A = \frac{4}{30} = \frac{2}{15}$

So $y_{p_2} = \frac{2}{15} e^{-5x}$

The particular solution to

$$y'' - y' = 20\sin(2x) + 4e^{-5x}$$

is

$$y_p = y_{p_1} + y_{p_2}$$

$$y_p = -4\sin(2x) + 2\cos(2x) + \frac{2}{15} e^{-5x}.$$

We can find this in one step by starting with the guess

$$y_p = A\sin(2x) + B\cos(2x) + Ce^{-5x}.$$

A Glitch!

$$y'' - y' = 3e^x$$

$g(x) = 3e^x$ a constant times e^x.

Guess $y_p = Ae^x$

$y_p' = Ae^x$, $y_p'' = Ae^x$

$y_p'' - y_p' = Ae^x - Ae^x = 3e^x$

$0Ae^x = 3e^x$ requires $0A = 3$

There's no value of A that satisfies $0A = 3$.

Consider the associated homogeneous eqn.

$$y'' - y' = 0$$

The solutions are $y_1 = e^x$ and $y_2 = 1$

so $y_c = C_1 e^x + C_2$.

Here $g(x)$ duplicates part of y_c, the complementary solution.

We'll consider cases

Using superposition as needed, begin with the assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

$$y'' - 2y' + y = -4e^x$$

Find y_c: solve $y'' - 2y' + y = 0$

Charc. eqn $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$ repeated root

$y_1 = e^x$, $y_2 = xe^x$, $y_c = C_1 e^x + C_2 x e^x$

Now find y_p: $g(x) = -4e^x$ guess $y_p = Ae^x$ ← part of y_c, won't work

Try again $y_p = Axe^x$ ← part of y_c, still won't work

Try again $y_p = Ax^2 e^x$ ← this is the correct form

Substitute into $y'' - 2y' + y = -4e^x$

$y_p = Ax^2 e^x$

$y_p' = Ax^2 e^x + 2Ax e^x$

$y_p'' = Ax^2 e^x + 2Ax e^x + 2Ax e^x + 2A e^x$

$= Ax^2 e^x + 4Ax e^x + 2A e^x$

$y_p'' - 2y_p' + y_p = -4 e^x$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2\left(Ax^2e^x + 2Axe^x\right) + Ax^2e^x = -4e^x$$

Collect like terms

$$x^2e^x\underbrace{(A-2A+A)}_{0} + xe^x\underbrace{(4A-4A)}_{0} + 2Ae^x = -4e^x$$

$$2Ae^x = -4e^x$$

$$\Rightarrow \quad 2A = -4 \quad \Rightarrow \quad A = -2$$

$$\boxed{\text{So} \quad y_p = -2x^2e^x}$$

The general solution to the nonhomogeneous equation is $y = y_c + y_p$

$$\boxed{y = c_1 e^x + c_2 x e^x - 2x^2 e^x}$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: $y'' - 4y' + 4y = 0$ $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \Rightarrow m = 2$ repeated root

$y_1 = e^{2x}$, $y_2 = xe^{2x}$

$y_c = C_1 e^{2x} + C_2 x e^{2x}$

For y_p, we consider y_{p_1} solving $y'' - 4y' + 4y = \sin(4x)$

and y_p solving $y'' - 4y' + 4y = xe^{2x}$

For y_{p_1}: $g_1(x) = \sin(4x)$

$y_{p_1} = A\sin(4x) + B\cos(4x)$ ← this is correct for y_{p_1}

For y_{p_2}: $g_2(x) = xe^{2x}$

1st guess $y_{p_2} = (Cx+D)e^{2x}$ ← Duplicates y_c

2nd guess $y_{p_2} = (Cx+D)xe^{2x} = (Cx^2+Dx)e^{2x}$ still duplicates

3rd guess $y_{p_2} = (Cx+D)x^2e^{2x} = (Cx^3+Dx^2)e^{2x}$ Bingo!

So $y_p = y_{p_1} + y_{p_2} = A\sin(4x) + B\cos(4x) + (Cx^3+Dx^2)e^{2x}$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c: $y''' - y'' + y' - y = 0$

Charc. Eqn $m^3 - m^2 + m - 1 = 0$

factor by grouping $m^2(m-1) + (m-1) = 0$

$(m^2+1)(m-1) = 0$

$\Rightarrow m^2 = -1$, $m = \pm\sqrt{-1} = \pm i$, $\alpha = 0$ $\beta = 1$

or $m = 1$

$y_1 = e^{0x}\cos(x)$, $y_2 = e^{0x}\sin(x)$, $y_3 = e^{x}$

$y_c = C_1 \cos x + C_2 \sin x + C_3 e^{x}$

Let $g_1(x) = \cos x$ and $g_2(x) = x^4$

For $g_1(x)$, guess

$$y_{p_1} = (A\cos x + B\sin x)x$$

$$= Ax\cos x + Bx\sin x$$ Correct form

For $g_2(x)$, guess

$$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$$ correct form

For the whole problem

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G.$$

Find the form of the particular soluition

$$y'' - 2y' + 5y = e^x + 7\sin(2x)$$

Find y_c: $y'' - 2y' + 5y = 0$, $m^2 - 2m + 5 = 0$

Complete the square

$m^2 - 2m + 1 + 4 = 0$

$(m-1)^2 + 4 = 0 \Rightarrow (m-1)^2 = -4$

$m - 1 = \pm\sqrt{-4} = \pm 2i$

$m = 1 \pm 2i$ $\alpha = 1$, $\beta = 2$

$y_1 = e^x \cos(2x)$ and $y_2 = e^x \sin(2x)$

$$y_c = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$$

For $g_1(x) = e^x$, guess $y_{p_1} = Ae^x$ ← works as written

$g_2(x) = 7\sin(2x)$

guess $y_{p_2} = B\sin(2x) + C\cos(2x)$

works as written

$$\boxed{\text{So } y_p = Ae^x + B\sin(2x) + C\cos(2x)}$$