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Section 9: Method of Undetermined Coefficients

Solve the IVP $\quad y'' - y = 4e^{-x} + 3 \quad y(0) = -1, \quad y'(0) = 1$

Find y_c: $\quad m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$ 2 distinct roots

$y_1 = e^x, \quad y_2 = e^{-x}, \quad y_c = C_1 e^x + C_2 e^{-x}$

Find y_p: Let $g_1(x) = 4e^{-x}$ and $g_2(x) = 3$

$y_{p_1} = Ae^{-x}$ won't work duplicates y_2

try $y_{p_1} = Axe^{-x}$ ← correct form

$y_{p_2} = B$ this is correct

Find y_{p_1} solving $y'' - y = 4e^{-x}$

$$y_{p_1} = Axe^{-x}$$
$$y_{p_1}' = Ae^{-x} - Axe^{-x}$$
$$y_{p_1}'' = -Ae^{-x} - Ae^{-x} + Axe^{-x}$$

$$y_{p_1}'' - y_{p_1} = -2Ae^{-x} + Axe^{-x} - Axe^{-x} = 4e^{-x}$$
$$-2Ae^{-x} = 4e^{-x}$$
$$-2A = 4$$
$$A = -2$$

$$\boxed{y_{p_1} = -2xe^{-x}}$$

Find y_{p_2} solving $y'' - y = 3$

$y_{p_2} = B$

$y_{p_2}' = 0$

$y_{p_2}'' = 0$

$y_{p_2}'' - y_{p_2} = 0 - B = 3$

$-B = 3 \Rightarrow B = -3$

$\boxed{y_{p_2} = -3}$

$y_p = y_{p_1} + y_{p_2} = -2xe^{-x} - 3$

The general solution to the ODE is

$$y = c_1 e^{x} + c_2 e^{-x} - 2x e^{-x} - 3$$

Apply $y(0) = -1$, $y'(0) = 1$

$$y' = c_1 e^{x} - c_2 e^{-x} - 2 e^{-x} + 2x e^{-x}$$

$$y(0) = c_1 e^{0} + c_2 e^{0} - 2\cdot 0 \cdot e^{0} - 3 = -1$$

$$c_1 + c_2 - 3 = -1 \implies c_1 + c_2 = 2$$

$$y'(0) = c_1 e^{0} - c_2 e^{0} - 2 e^{0} + 2\cdot 0 \cdot e^{0} = 1$$

$$c_1 - c_2 - 2 = 1 \implies c_1 - c_2 = 3$$

$$\begin{array}{l} c_1 + c_2 = 2 \\ c_1 - c_2 = 3 \\ \hline 2c_1 \quad = 5 \end{array} \quad \text{add} \quad \Longrightarrow \quad c_1 = \frac{5}{2}$$

$$c_2 = 2 - c_1 = \frac{4}{2} - \frac{5}{2} = \frac{-1}{2}$$

$$\boxed{\begin{array}{l}\text{The solution to the IVP is} \\[1em] y = \frac{5}{2}e^{x} - \frac{1}{2}e^{-x} - 2xe^{-x} - 3\end{array}}$$

Section 10: Variation of Parameters

The Method of Undetermined Coefficients require our DE to have two critical properties: **(1)** The left side MUST be constant ceofficient, and **(2)** the right side MUST come from the restricted class of functions (poly., exp., sine/cosine, their sums or products).

Consider the equation $y'' + y = \tan x$. What happens if we try to find a particular solution having the same *form* as the right hand side?

We need all like terms that arise when taking derivatives.

Suppose $y_p = A\tan x$

$y_p' = A\sec^2 x \leftarrow$ $\sec^2 x$ isn't accounted for

Try again

$$y_p = A\tan x + B\sec^2 x$$

$$y_p' = A\sec^2 x + \underbrace{2B\sec^2 x\tan x}_{\text{not accounted for}}$$

Try $y_p = A\tan x + B\sec^2 x + C\sec^2 x\tan x$

$$y_p' = A\sec^2 x + 2B\sec^2 x\tan x + 2C\sec^2 x\tan^2 x + C\sec^4 x$$

Undetermined coef. isn't feasible.

Consider the equation $x^2y'' + xy' - 4y = e^x$. What happens if we assume $y_p = Ae^x$?

$y_p' = Ae^x$

$y_p'' = Ae^x$

$$x^2 y_p'' + x y_p' - 4y_p = e^x$$

$$Ax^2e^x + Axe^x - 4Ae^x = e^x$$

we can't match because the coefficients on the left create additional "like terms."

We need another method!

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

We'll need a second equation.

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We'll set

$$u_1'y_1 + u_2'y_2 = 0$$

$$y_p' = u_1y_1' + u_2y_2' + \underbrace{u_1'y_1 + u_2'y_2}_{=0}$$

$$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$y_p'' + P(x)y_p' + Q(x)y_p = g(x)$$

$$u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' + P(x)\left(u_1y_1' + u_2y_2'\right) + Q(x)\left(u_1y_1 + u_2y_2\right) = g(x)$$

Collect u_1, u_2, u_1' and u_2' terms.

$$u_1\underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}_{0} + u_2\underbrace{\left(y_2'' + P(x)y_2' + Q(x)y_2\right)}_{0} + u_1'y_1' + u_2'y_2' = g(x)$$

Since y_1, y_2 solve the homogeneous eqn.

$$u_1'y_1' + u_2'y_2' = g(x)$$

We have 2 eqns for u_1 and u_2

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(x)$$

In a matrix format this can be written as

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

We'll solve using Crammer's rule

Let $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}$ and $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}$

Let $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ (the Wronskian of y_1 and y_2)

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

$$u_1 = \int \frac{-g(x)\, y_2(x)}{W}\, dx$$

$$u_2 = \int \frac{y_1(x)\, g(x)}{W}\, dx$$

and so $\quad y_p = u_1 y_1 + u_2 y_2$

Example:

Solve the ODE $y'' + y = \tan x$.

It's in standard form

$g(x) = \tan x$

Find y_1, y_2: $\quad y'' + y = 0$

Char. eqn $\quad m^2 + 1 = 0 \Rightarrow m^2 = -1, \quad m = \pm i$

$\alpha = 0, \ \beta = 1$

$y_1 = e^{0x} \cos x = \cos x$

$y_2 = e^{0x} \sin x = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$g(x) = \tan x$, $y_1 = \cos x$, $y_2 = \sin x$, $W = 1$

$$u_1 = \int \frac{-g(x)\,y_2(x)}{W}\,dx = \int \frac{-\tan x \sin x}{1}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{(1-\cos^2 x)}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\,dx = -\ln|\sec x + \tan x| + \sin x$$

$$u_1 = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{g(x)\, y_1(x)}{W}\, dx = \int \frac{\tan x \, \cos x}{1}\, dx$$

$$= \int \sin x \, dx = -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2 \Rightarrow$$

$$y_p = (\sin x - \ln|\sec x + \tan x|)\cos x - \cos x \sin x$$

$$y_p = \sin x \cos x - \cos x \ln|\sec x + \tan x| - \cos x \sin x$$

$$y_p = -\cos x \ln|\sec x + \tan x|$$

The general solution is

$$\boxed{y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|}$$

Example:

Solve the ODE

$$y'' - 2y' + y = \frac{e^x}{1+x^2}.$$

$$g(x) = \frac{e^x}{1+x^2}$$

Find y_c: $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$ repeated root

$$y_1 = e^x, \quad y_2 = xe^x, \quad y_c = c_1 e^x + c_2 x e^x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x(xe^x + e^x) - e^x(xe^x) = e^{2x}$$

$$g(x)=\frac{e^x}{1+x^2},\quad y_1=e^x,\quad y_2=xe^x,\quad W=e^{2x}$$

$$u_1=\int\frac{-g(x)\,y_2(x)}{W}\,dx=\int\frac{-\frac{e^x}{1+x^2}\left(xe^x\right)}{e^{2x}}\,dx=-\int\frac{xe^{2x}}{e^{2x}(1+x^2)}\,dx$$

$$=-\int\frac{x}{1+x^2}\,dx \qquad \text{Let } v=1+x^2,\quad dv=2x\,dx,\quad \tfrac{1}{2}dv=x\,dx$$

$$=-\frac{1}{2}\int\frac{dv}{v}=-\frac{1}{2}\ln|v|=-\frac{1}{2}\ln(1+x^2)$$

$$u_2=\int\frac{g(x)\,y_1(x)}{W}\,dx=\int\frac{\frac{e^x}{1+x^2}\cdot e^x}{e^{2x}}\,dx=\int\frac{e^{2x}}{e^{2x}(1+x^2)}\,dx$$

$$= \int \frac{1}{1+x^2}\,dx = \tan^{-1}x$$

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{2}\ln(1+x^2)\,e^{x} + x e^{x}\tan^{-1}x$$

The general solution to the ODE is

$$y = c_1 e^{x} + c_2 x e^{x} - \frac{1}{2} e^{x} \ln(1+x^2) + x e^{x}\tan^{-1}x$$

Example:

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1 x^2 + c_2 x^{-2}$ is the complementary solution.

From y_c, we have $y_1 = x^2$ and $y_2 = x^{-2}$

Standard form: $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}$

$$g(x) = \frac{\ln x}{x^2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^2x^{-3} - 2xx^{-2} = -4x^{-1}$$

$$g(x) = \frac{\ln x}{x^2}, \quad y_1 = x^2, \quad y_2 = x^{-2}, \quad W = -4x^{-1}$$

$$u_1 = \int \frac{-g(x)y_2(x)}{W}\,dx = \int \frac{-\frac{\ln x}{x^2}\cdot x^{-2}}{-4x^{-1}}\,dx = \int \frac{x \ln x}{4x^4}\,dx$$

$$= \frac{1}{4}\int \frac{\ln x}{x^3}\,dx$$

By parts $\quad w = \ln x \quad dw = \frac{1}{x}\,dx$

$v = \frac{x^{-2}}{-2} \quad dv = x^{-3}\,dx$

$$= \frac{1}{4}\left[\frac{-1}{2x^2}\ln x - \int \frac{-1}{2x^2}\cdot\frac{1}{x}\,dx\right]$$

$$= \frac{1}{4}\left[\frac{-1}{2x^2}\ln x + \frac{1}{2}\int x^{-3}\,dx\right] = \frac{1}{4}\left[\frac{-1}{2x^2}\ln x + \frac{1}{2}\left(\frac{x^{-2}}{-2}\right)\right]$$

$$u_1 = \frac{-1}{8x^2}\ln x - \frac{1}{16x^2}$$

$$u_2 = \int \frac{g(x)y_1(x)}{W}\,dx = \int \frac{\frac{\ln x}{x^2}\quad x^2}{-4x^{-1}}\,dx = \frac{-1}{4}\int x\ln x\,dx$$

By parts $\quad w = \ln x \quad dw = \frac{1}{x}\,dx$

$v = \frac{x^2}{2} \quad dv = x\,dx$

$$u_2 = \frac{-1}{4}\left[\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx\right] = \frac{-1}{4}\left[\frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx\right]$$

$$u_2 = \frac{-1}{4}\left[\frac{x^2}{2}\ln x - \frac{1}{2}\,\frac{x^2}{2}\right] = \frac{-x^2}{8}\ln x + \frac{x^2}{16}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{-1}{8x^2}\ln x - \frac{1}{16x^2}\right)x^2 + \left(\frac{-x^2}{8}\ln x + \frac{x^2}{16}\right)x^{-2}$$

$$= \frac{-1}{8}\ln x - \frac{1}{16} - \frac{1}{8}\ln x + \frac{1}{16} = \frac{-1}{4}\ln x$$

The general solution is $\boxed{y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4}\ln x}$

Solve the IVP

$$x^2y'' + xy' - 4y = \ln x, \quad y(1) = -1, \quad y'(1) = 0$$

From the last example

$$y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4}\ln x$$

$$y' = 2c_1 x - 2c_2 x^{-3} - \frac{1}{4x}$$

$$y(1) = c_1(1)^2 + c_2(1)^{-2} - \frac{1}{4}\ln 1 = -1 \Rightarrow \quad c_1 + c_2 = -1$$

$$y'(1) = 2c_1 \cdot 1 - 2c_2(1)^{-3} - \frac{1}{4\cdot 1} = 0 \Rightarrow 2c_1 - 2c_2 = \frac{1}{4}$$

$$2c_1 + 2c_2 = -2$$

$$2c_1 - 2c_2 = 1/4$$

add

$$4c_1 = -2 + \frac{1}{4} = \frac{-7}{4} \implies c_1 = \frac{-7}{16}$$

From $c_1 + c_2 = -1$, $\quad c_2 = -1 - c_1 = -1 + \frac{7}{16} = \frac{-16+7}{16} = \frac{-9}{16}$

The soln. to the IVP is

$$\boxed{y = \frac{-7}{16} x^2 - \frac{9}{16} x^{-2} - \frac{1}{4} \ln x}$$

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion.**

Harmonic Motion gif

Building an Equation: Hooke's Law

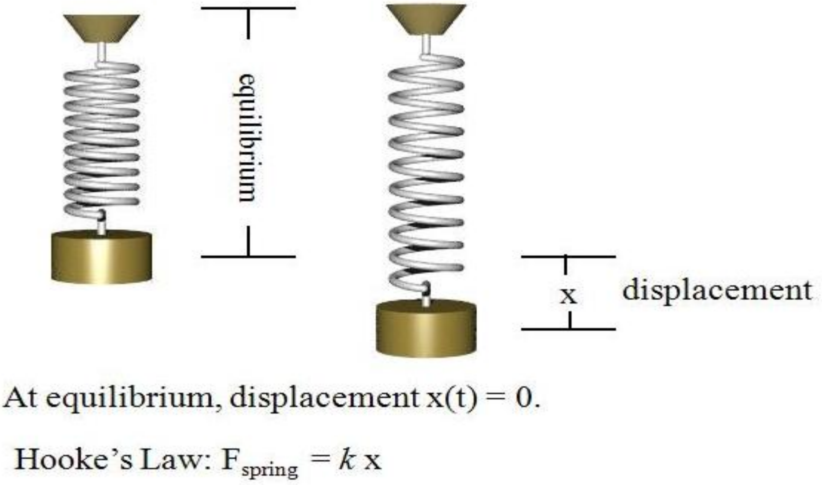

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ Force = mass times acceleration

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ Force exerted by the spring is proportional to displacement
The force imparted by the spring opposes the direction of motion.

$$m\frac{d^2x}{dt^2} = -kx \quad \Longrightarrow \quad x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Convention We'll Use: Up will be positive ($x > 0$), and down will be negative ($x < 0$). This orientation is arbitrary and follows the convention in Trench.