June 6 Math 2306 sec 52 Summer 2016

Section 2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation on some interval I containing x_0.

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an ***initial value problem*** **(IVP)**.
Note that all conditions are given at the same input value x_0, and the number of initial conditions is equal to the order of the ODE.

Example (Part 1)

Verify that $y = ce^{-x^2}$ is a solution of the ODE for any constant c.

$$\frac{dy}{dx} + 2xy = 0$$

$y = ce^{-x^2}$, $\quad y' = ce^{-x^2} \cdot (-2x)$

$= -2xce^{-x^2}$

$$\frac{dy}{dx} + 2xy \stackrel{?}{=} 0$$

$$-2xce^{-x^2} + 2x\left(ce^{-x^2}\right) \stackrel{?}{=} 0$$

$$-2xce^{-x^2} + 2xce^{-x^2} \stackrel{?}{=} 0$$

$$0 = 0 \quad \text{an identity}$$

So $y = ce^{-x^2}$ solves the ODE.

Example (Part 2)

Solve the IVP

$$\frac{dy}{dx}+2xy=0, \quad y(0)=3$$

$y=Ce^{-x^2}$ solves the ODE for any value of C.

Impose $y(0)=3$

$$3=Ce^{-0^2}=C \implies C=3$$

The solution to the IVP is $y=3e^{-x^2}$

Graphical Interpretation

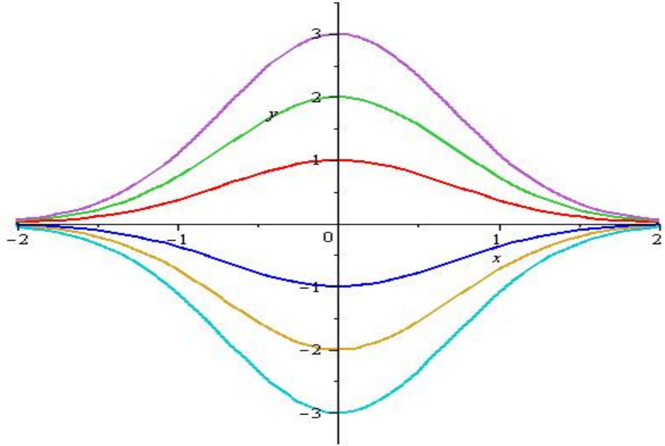

Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

Example (Part 1)

Show that $x = c_1 \cos(2t) + c_2 \sin(2t)$ is a 2-parameter family of solutions of the ODE $x'' + 4x = 0$.

$$x = c_1 \cos(2t) + c_2 \sin(2t)$$

$$x' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$$

$$x'' = -4c_1 \cos(2t) - 4c_2 \sin(2t)$$

$$x'' + 4x \stackrel{?}{=} 0$$

$$-4c_1 \cos(2t) - 4c_2 \sin(2t) + 4\left(c_1 \cos(2t) + c_2 \sin(2t)\right) \stackrel{?}{=} 0$$

$$-4c_1 \cos(2t) - 4c_2 \sin(2t) + 4c_1 \cos(2t) + 4c_2 \sin(2t) \stackrel{?}{=} 0$$

$$\cos(2t)\left(-4c_1+4c_1\right)+\sin(2t)\left(-4c_2+4c_2\right)\stackrel{?}{=}0$$

$$0+0=0 \quad \checkmark \quad \text{an identity}$$

So $x=c_1\cos(2t)+c_2\sin(2t)$

is a 2-parameter family of

solutions to the ODE.

Example (Part 2)

Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

Recall $\cos(\pi) = -1$ and $\sin(\pi) = 0$

$x = c_1\cos(2t) + c_2\sin(2t)$

$x' = -2c_1\sin(2t) + 2c_2\cos(2t)$

$x\left(\frac{\pi}{2}\right) = c_1\cos\left(2\cdot\frac{\pi}{2}\right) + c_2\sin\left(2\cdot\frac{\pi}{2}\right) = -1$

$x'\left(\frac{\pi}{2}\right) = -2c_1\sin\left(2\cdot\frac{\pi}{2}\right) + 2c_2\cos\left(2\cdot\pi/2\right)$

$c_1(-1) + c_2(0) = -1$

$c_1 = 1$

$4 = -2c_1\cdot 0 + 2c_2\cdot(-1)$

$4 = -2c_2 \Rightarrow c_2 = -2$

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

Which is the correct solution?

(a) $x(t) = -\cos(2t) + 4\sin(2t)$

(b) $x(t) = -\cos(2t) - 2\sin(2t)$

(c) $x(t) = \cos(2t) - 2\sin(2t)$ ← the correct answer

$x = c_1 \cos(2t) + c_2 \sin(2t)$

where $c_1 = 1$, $c_2 = -2$

(d) $x(t) = \cos(2t) + 4\sin(2t)$

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\left(\frac{dy}{dx}\right)^2 + 1 = -y^2$.

The left is 1 or bigger, the right is zero or smaller.

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

Lets show that $y = \frac{x^4}{16}$ solves $\frac{dy}{dx} = x\sqrt{y}$

and solves $y(0) = 0$.

Does $y(0) = 0$? $\quad y(0) = \frac{0^4}{16} = 0 \quad$ yes!

Does it solve the ODE? $\quad y = \frac{1}{16}x^4, \quad y' = \frac{4}{16}x^3 = \frac{1}{4}x^3$

$$\frac{dy}{dx} \overset{?}{=} x\sqrt{y}$$

$$\frac{1}{4}x^3 \overset{?}{=} x\sqrt{\frac{x^4}{16}}$$

$$\frac{1}{4}x^3 \overset{?}{=} x\,\frac{x^2}{4}$$

$$\frac{1}{4}x^3 = \frac{1}{4}x^3$$

an identity

yes, it solves the ODE too.

So $y = \frac{x^4}{16}$ solves the IVP.

Let's find another by "guessing."

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0)=0$$

We could guess a constant solution $y = c$

If $y(0) = c$ and $y(0)=0$ we need $c=0$.

If $y=0$, then $\frac{dy}{dx} = 0$ so

$$\frac{dy}{dx} \stackrel{?}{=} x\sqrt{y}$$

$$0 \stackrel{?}{=} x\sqrt{0}$$

$$0 = 0$$

The constant function $y=0$ solves the IVP too.

This is the <u>trivial</u> solution.

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$y = \int y' \, dx$

so $y = \int (4e^{2x} + 1)\, dx$

$= 4\frac{e^{2x}}{2} + x + C \Rightarrow \boxed{y = 2e^{2x} + x + C}$

A 1-parameter family of solutions.

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

(Note this is a **product** of a function of only x and one of only y.)

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\frac{dy}{dx} = x^3 y$

yes it's separable

$g(x) = x^3$ and $h(y) = y$

(b) $\frac{dy}{dx} = 2x + y$

no, this is not separable

(c) $\dfrac{dy}{dx} = \sin(xy^2)$ No, not separable.

The sine can't be written as a product like $g(x) \cdot h(y)$

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$ $\Rightarrow$ $\dfrac{dy}{dt} = te^{t-y} = te^{t} \cdot e^{-y} = \dfrac{te^{t}}{e^{y}}$

Yes, it's separable with $g(t) = te^{t}$

$h(y) = e^{-y}$

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

$$y + C_1 = G(x) + C_2$$

where $G(x)$ is any antiderivative of $g(x)$

$$y = G(x) + C \qquad (\text{where } C = C_2 - C_1)$$

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$*$ Recall $\quad dy = \frac{dy}{dx}\,dx$ $*$

$$\frac{dy}{dx} = g(x)h(y)$$

$$\Longrightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\Longrightarrow \quad p(y)\frac{dy}{dx}\,dx = g(x)\,dx$$

$$p(y)\,dy = g(x)\,dx$$

Now we integrate each side

$$\int p(y)\,dy = \int g(x)\,dx$$

$$P(y) = G(x) + C$$

where $P(y)$ and $G(x)$ are antiderivatives of $p(y)$ and $g(x)$, respectively.

$P(y) = G(x) + C$ defines a 1-parameter family implicitly.

Solve the ODE

$$\frac{dy}{dt} - te^{t-y} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = te^{t} \cdot e^{-y}$$

$$\frac{1}{e^{-y}} \frac{dy}{dt} = te^{t} \quad \Rightarrow \quad e^{y} \frac{dy}{dt} dt = te^{t} dt$$

$$\Rightarrow \quad e^{y} dy = te^{t} dt$$

$$\int e^{y} dy = \int te^{t} dt$$

by parts on the right

$u = t, \quad du = dt$

$v = e^{t} \quad dv = e^{t} dt$

$$e^{y} = te^{t} - \int e^{t} dt$$

$$e^y = te^t - e^t + C$$

This gives a one-parameter family of solutions implicitly.

Solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y} = -x\left(\frac{1}{y}\right) \Rightarrow \frac{1}{1/y}\frac{dy}{dx} = -x$$

$$y\frac{dy}{dx}\,dx = -x\,dx$$

$$y\,dy = -x\,dx \Rightarrow \int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C \Rightarrow x^2 + y^2 = k \quad \text{where } k = 2C$$

Solve the IVP[1]

$$\frac{dQ}{dt} = -2(Q - 70), \quad Q(0) = 180$$

$$\frac{dQ}{dt} = -2(Q-70) \qquad \text{assuming } Q-70 \neq 0$$

$$\frac{1}{Q-70}\frac{dQ}{dt} = -2 \quad \Rightarrow \quad \frac{1}{Q-70}\frac{dQ}{dt}\,dt = -2\,dt$$

$$\frac{1}{Q-70}\,dQ = -2\,dt$$

$$\int \frac{1}{Q-70}\,dQ = \int -2\,dt$$

[1] Recall IVP stands for *initial value problem*.

$$\Rightarrow \ln|Q-70| = -2t + C$$

Let's exponentiate both sides

$$e^{\ln|Q-70|} = e^{-2t+C} = e^{C} e^{-2t}$$

$$|Q-70| = e^{C} e^{-2t}$$ Let $k = \pm e^{C}$ or zero

$$Q-70 = k e^{-2t}$$

$$\Rightarrow Q = 70 + k e^{-2t}$$

Now impose $Q(0) = 180$

$$180 = 70 + ke^{-2\cdot 0} = 70 + k$$

$$\Rightarrow \quad k = 110$$

The solution to the IVP is

$$\boxed{Q = 70 + 110\, e^{-2t}.}$$

Caveat regarding division by $h(y)$.

Solve the IVP by separation of variables[2]

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

$$\frac{dy}{dx} = x\,y^{1/2} \quad \Rightarrow \quad \frac{1}{y^{1/2}}\frac{dy}{dx} = x\,dx$$

$$y^{-1/2}\frac{dy}{dx}\,dx = x\,dx \quad \Rightarrow \quad y^{-1/2}\,dy = x\,dx$$

$$\int y^{-1/2}\,dy = \int x\,dx \quad \Rightarrow \quad \frac{y^{1/2}}{1/2} = \frac{x^2}{2} + C$$

[2] Remember that one solution is $y(x) = 0$ (for all x).

$$y^{1/2} = \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{2}\cdot C \implies y^{1/2} = \frac{x^2}{4} + k$$

where $k = \frac{1}{2}C$

$$y = \left(\frac{x^2}{4} + k\right)^2$$ a 1-parameter family of solutions to the ODE.

Impose $y(0) = 0$

$$0 = \left(\frac{0^2}{4} + k\right)^2 = k^2 \implies k = 0$$

So the solution to the IVP is

$$y = \left(\frac{x^2}{4} + 0\right)^2 = \left(\frac{x^2}{4}\right)^2 = \frac{x^4}{16}$$

$$\boxed{y = \frac{x^4}{16}}$$

* we know that $y = 0$ is also a solution to the IVP *

This is not a member of $y = \left(\frac{x^2}{4} + k\right)^2$

Losing a solution

The previous example illustrates that it is possible to *lose* a solution. This is something we can be aware of.

For the IVP $\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$, we could ask the question

$$\text{Is } f(x, y_0) = 0?$$

If the answer is **yes**, then the constant function $y(x) = y_0$ is a solution of the IVP.

If $f(x,y_0)=0$ and $y(x)=y_0$ then $\frac{dy}{dx}=0$ and the ODE is $0=0$ which is always true.

Find a solution of the IVP

$$\frac{dy}{dx} = y^2 - 4, \quad y(0) = -2$$

A solution to the IVP is $y(x) = -2$.

Verify: If $y(x) = -2$ then $y(0) = -2$.

the initial condition holds.

If $y(x) = -2$ then $\frac{dy}{dx} = 0$

$\frac{dy}{dx} \stackrel{?}{=} y^2 - 4$, $\quad 0 \stackrel{?}{=} (-2)^2 - 4$

$0 \stackrel{?}{=} 4 - 4 \Rightarrow 0 = 0$ ✓

So $y(x) = -2$ solves the ODE too.

Hence $y(x) = -2$ solves the IVP.

Solve the ODE by separation of variables

$$\frac{dy}{dx} = y^2 - 4$$

If $y^2 - 4 \neq 0$ then

$$\frac{1}{y^2-4}\frac{dy}{dx} = 1$$

$$\frac{1}{y^2-4}\frac{dy}{dx}\,dx = dx$$

$$\frac{1}{y^2-4}\,dy = dx$$

$$\int \frac{1}{y^2-4}\,dy = \int dx$$

Use a partial fraction decomp on the left. The problem is left to the reader.

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus)

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

we have $\frac{dy}{dt} = g(t)$ so

$$\int_{x_0}^{x} \frac{dy}{dt}\,dt = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y_0 = \int_{x_0}^{x} g(t)\,dt$$

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt}$$

This is the solution to the IVP.

Let's verify that this solves the IVP.

Does it satisfy $y(x_0) = y_0$?

$$y(x_0) = y_0 + \int_{x_0}^{x_0} g(t)\,dt = y_0 + 0 = y_0$$

yes

Does it solve $\frac{dy}{dx} = g(x)$

$$\frac{d}{dx}\, y(x) \overset{?}{=} \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} g(t)\,dt \right)$$

$$\frac{dy}{dx} \overset{?}{=} 0 + \frac{d}{dx}\int_{x_0}^{x} g(t)\,dt$$

$$\overset{?}{=} 0 + g(x)$$

$\frac{dy}{dx} = g(x)$ so yes it solves the ODE.

Example: Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

Note $\int \sin(x^2)\,dx$ doesn't have an elementary antiderivative,

Here $g(x) = \sin(x^2)$ so $g(t) = \sin(t^2)$

$x_0 = \sqrt{\pi}$ and $y_0 = 1$

The solution is $\boxed{y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt}$

From before

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$