June 8 Math 2306 sec 52 Summer 2016

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$P(x) = \frac{a_0(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)y = f(x). \qquad f(x) = \frac{g(x)}{a_1(x)}$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the equation
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy = e^x$$

By the product rule

$$\frac{d}{dx}\left[x^2 y\right] = x^2\frac{dy}{dx} + 2xy$$

So our equation is

$$\frac{d}{dx}\left[x^2 y\right] = e^x$$

* our goal is to find y *

Integrate both sides

$$\int \frac{d}{dx}\left[x^2 y\right]dx = \int e^x\,dx$$

$$x^2 y = e^x + C$$

Divide by x^2 to isolate y

$$y = \frac{e^x + C}{x^2}$$

$$y = \frac{e^x}{x^2} + \frac{C}{x^2}$$

$$y_p = \frac{e^x}{x^2} \quad \text{and} \quad y_c = \frac{C}{x^2}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

- Based on the previous example, we want the left side to collapse as one derivative term (a product rule)

- Multiply both sides by some function $\mu(x)$ such that the left side is our product rule.

$$\frac{dy}{dx} + P(x)\, y = f(x)$$

assume $\mu(x)$ exists.

$$\mu(x) \frac{dy}{dx} + P(x)\mu(x)\, y = \mu(x) f(x)$$

The left needs to be $\frac{d}{dx}\left[\mu(x)\, y\right]$

$$\frac{d}{dx}\left[\mu(x)\, y\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}\, y$$

Match to $\mu(x)\frac{dy}{dx} + P(x)\mu(x)\, y$

This requires $\quad \frac{d\mu}{dx} y = P(x)\mu(x) y$

For $y \neq 0$, μ solves the separable equation

$$\frac{d\mu}{dx} = P(x)\mu$$

Separate variables $\quad \frac{1}{\mu}\frac{d\mu}{dx} = P(x)$

$$\frac{1}{\mu}\frac{d\mu}{dx}\,dx = P(x)\,dx$$

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\, dx \qquad \text{Lets assume } \mu > 0$$

$$\ln \mu = \int P(x)\, dx$$

$$\Rightarrow \boxed{\mu = e^{\int P(x)\,dx}}$$

This is the function needed to get a product rule on the left side. It's called an

<u>integrating factor.</u>

So
$$\mu \frac{dy}{dx} + P(x)\mu y = \mu(x) f(x)$$

$$\frac{d}{dx}\left[\mu y\right] = \mu(x) f(x)$$

$$\int \frac{d}{dx}\left[\mu y\right] dx = \int \mu(x) f(x)\, dx$$

$$\mu y = \int \mu(x) f(x)\, dx + C$$

$$\boxed{y = \underbrace{\frac{1}{\mu}\int f(x)\mu(x)\, dx}_{y_p} + \underbrace{\frac{C}{\mu}}_{y_c}}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Solve the ODE

Let's assume that $x > 0$

$$x^2\frac{dy}{dx}+2xy = e^x$$

Standard form (divide by x^2)

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{e^x}{x^2} \qquad P(x) = \frac{2}{x}$$

Build $\mu = e^{\int P(x)\,dx}$

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx = 2\ln x = \ln x^2$$

So $\mu = e^{\ln x^2} = x^2$

Mult. standard form by $\mu = x^2$

$$x^2\left(\frac{dy}{dx} + \frac{2}{x}y\right) = x^2\left(\frac{e^x}{x^2}\right)$$

$$\frac{d}{dx}\left[x^2 y\right] = e^x$$

$$\int \frac{d}{dx}\left[x^2 y\right]dx = \int e^x\,dx$$

$$x^2 y = e^x + C$$

$$\Rightarrow \boxed{y = \frac{e^x + C}{x^2}}$$

Solve the ODE

Find the general solution using an integrating factor. Then verify that your result actually solves the ODE.

$$\frac{dy}{dx}+y=3xe^{-x}$$

The eqn is in standard form.

$P(x)=1$

Build μ: $\mu = e^{\int P(x)\,dx}$ $\qquad \int P(x)\,dx = \int dx = x$

$\Rightarrow \quad \mu = e^{x}$

Mult. eqn by μ $\qquad e^{x}\left(\frac{dy}{dx}+y\right) = e^{x}\left(3xe^{-x}\right)$

$$\frac{d}{dx}\left[e^x y\right] = 3x$$

Note

$$e^x\left(\frac{dy}{dx}+y\right) = e^x\frac{dy}{dx}+e^x y$$

$$\int \frac{d}{dx}\left[e^x y\right]dx = \int 3x\,dx$$

$$e^x y = \frac{3x^2}{2} + C$$

$$y = \frac{\frac{3x^2}{2}+C}{e^x} = \frac{3}{2}x^2 e^{-x} + Ce^{-x}$$

Our solution is $\boxed{y = \frac{3}{2}x^2 e^{-x} + Ce^{-x}.}$

Let's verify that $y = \frac{3}{2}x^2e^{-x} + Ce^{-x}$ solves $\frac{dy}{dx} + y = 3xe^{-x}$

$$\frac{dy}{dx} = \frac{3}{2}(2x)e^{-x} + \frac{3}{2}x^2(-e^{-x}) + C(-e^{-x})$$

$$\frac{dy}{dx} = 3xe^{-x} - \frac{3}{2}x^2e^{-x} - Ce^{-x}$$

$$\frac{dy}{dx} + y \stackrel{?}{=} 3xe^{-x}$$

$$3xe^{-x} - \cancel{\frac{3}{2}x^2e^{-x}} - \cancel{Ce^{-x}} + \cancel{\frac{3}{2}x^2e^{-x}} + \cancel{Ce^{-x}} \stackrel{?}{=} 3xe^{-x}$$

$$3xe^{-x} = 3xe^{-x}$$ an identity.

Hence our solution is a solution.

Solve the Initial Value Problem

Find the solution of the IVP.

$$\cos\theta\frac{dy}{d\theta}+\sin\theta y = 1, \quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}, \quad y(0)=-2$$

Standard form

$$\frac{dy}{d\theta}+\frac{\sin\theta}{\cos\theta}y=\frac{1}{\cos\theta}$$

$$\frac{dy}{d\theta}+\tan\theta\, y=\sec\theta \qquad P(\theta)=\tan\theta$$

Find $\mu = e^{\int P(\theta)d\theta}$

$$\int P(\theta)\,d\theta = \int \tan\theta\, d\theta = \ln|\sec\theta|$$

$$= \ln(\sec\theta) \qquad \sec\theta>0 \text{ for } -\pi/2<\theta<\pi/2$$

So

$$\mu = e^{\ln(\sec\theta)} = \sec\theta$$

Mult. eqn in standard form by μ

$$\sec\theta\left(\frac{dy}{d\theta} + \tan\theta\, y\right) = \sec\theta \sec\theta$$

$$\frac{d}{d\theta}\left[\sec\theta\, y\right] = \sec^2\theta$$

$$\int \frac{d}{d\theta}\left[\sec\theta\, y\right] d\theta = \int \sec^2\theta\, d\theta$$

$$\sec\theta\, y = \tan\theta + C$$

$$y = \frac{\tan\theta + C}{\sec\theta} = \cos\theta\tan\theta + C\cos\theta$$

$$\Rightarrow \quad y = \sin\theta + C\cos\theta$$

our general soln. to the ODE.

Impose the initial condition $y(0) = -2$

$$-2 = \sin 0 + C\cos 0 = 0 + C\cdot 1$$

$$C = -2$$

The solution to the IVP is $y = \sin\theta - 2\cos\theta$.

Find the General Solution

Find the general solution of the ODE.

$$x\frac{dy}{dx} - y = 2x^2$$

Let's assume $x > 0$.

Standard form $\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x$

$P(x) = \frac{-1}{x}$, $\mu = e^{\int P(x)\,dx}$ $\quad \int P(x)\,dx = \int \frac{-1}{x}\,dx = -\ln x$

$$\mu = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = 2x \cdot x^{-1}$$

$$\frac{d}{dx}\left[x^{-1}y\right] = 2$$

$$\int \frac{d}{dx}\left[x^{-1}y\right] dx = \int 2\, dx$$

$$x^{-1}y = 2x + C$$

$$y = \frac{2x+C}{x^{-1}} = 2x^2 + Cx$$

$$x\frac{dy}{dx} - y = 2x^2$$

Which is the correct **general solution**?

(a) $y = \frac{2}{3}x^4 + Cx$

(b) $y = 2x^2$

(c) $y = x + \frac{C}{x}$

(d) $y = 2x^2 + Cx$ $\leftarrow$ the correct solution.

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

* there's a typo here

$y_p = \frac{3}{2}x^2 e^{-x}$

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2 + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2 \quad \text{and} \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**