June 11 Math 2254 sec 001 Summer 2015

Section 7.1: Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

The integration by parts formula can be restated as

$$\int u\,dv = uv - \int v\,du$$

$$\int u \, dv = uv - \int v \, du$$

Evaluate

$$\int x \cos x \, dx$$

We completed this problem by setting $u = x$ and $dv = \cos x \, dx$. This was based on two primary observations:

(1) we can find $v = \sin x$ from dv (we know the anti-derivative of $\cos x$), and

(2) $du = dx$ which made the last integral $\int v \, du$ easier to evaluate.

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$$

Show that the choice $u = \cos x$ and $dv = x\, dx$ would not work so well.

$$\int x \cos x\, dx$$

If $\quad u = \cos x \qquad du = -\sin x\, dx$

$\qquad\qquad v = \dfrac{x^2}{2} \qquad dv = x\, dx$

$$\int x \cos x\, dx = \frac{x^2}{2} \cos x \;-\; \int \frac{x^2}{2}(-\sin x)\, dx$$

$$= \frac{x^2}{2} \cos x \;+\; \int \frac{x^2}{2} \sin x\, dx$$

This is tougher than the integral we started with.

Usually (not always) we take x to be the power function.

$\int u\,dv = uv - \int v\,du$

Evaluate
$\int y^2 e^y\,dy$

Let $\quad u = y^2 \qquad du = 2y\,dy$

$\qquad\qquad v = e^y \qquad dv = e^y\,dy$

Use Int. by parts
again

$\int y^2 e^y\,dy = y^2 e^y - \int 2y\, e^y\,dy$

Let $\quad u = 2y \quad du = 2\,dy$

$\qquad v = e^y \quad dv = e^y\,dy$

$\qquad\qquad = y^2 e^y - \left(2y\, e^y - \int 2 e^y\,dy \right)$

$$= y^2 e^y - 2y\, e^y + \int 2e^y\, dy$$

$$= y^2 e^y - 2y\, e^y + 2e^y + C$$

$$\int u\,dv = uv - \int v\,du$$

Evaluate

$$\int x^3 \ln x\,dx$$

Let $\quad u = x^3 \qquad du = 3x^2\,dx$

$\qquad v = ? \qquad dv = \ln x\,dx$

we don't know!!

Try again:

let $\quad u = \ln x \qquad du = \frac{1}{x}\,dx$

$\qquad v = \frac{x^4}{4} \qquad dv = x^3\,dx$

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^4}{4} \ln x - \int \frac{x^3}{4} \, dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \cdot \frac{x^4}{4} + C$$

$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$$

Something a little different (int. by parts)

Evaluate
$$\int \sin^{-1} x \, dx$$

Let $\quad u = \sin^{-1} x \qquad du = \dfrac{1}{\sqrt{1-x^2}} \, dx$

$$v = x \qquad dv = dx$$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int x \, \frac{1}{\sqrt{1-x^2}} \, dx$$

We'll use
substitution
on last part

Let $u = 1 - x^2$

$du = -2x \, dx$

$-\frac{1}{2} \, du = x \, dx$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \frac{-1}{2} \int \frac{1}{\sqrt{u}} \, du$$

$$= x \sin^{-1} x + \frac{1}{2} \int u^{-1/2} \, du$$

$$= x \sin^{-1} x + \frac{1}{2} \frac{u^{1/2}}{1/2} + C$$

$$= x \sin^{-1} x + \sqrt{1 - x^2} + C$$

Another unusual use of Int. by Parts

Evaluate the integral $\mathcal{I}$ where

$$\mathcal{I} = \int e^x \cos x \, dx$$

Let $u = e^x \qquad du = e^x \, dx$

$ v = \sin x \qquad dv = \cos x \, dx$

$$I = e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x \sin x - \left(-e^x \cos x - \int e^x (-\cos x) dx \right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

Int. by parts again

$u = e^x \qquad du = e^x \, dx$

$v = -\cos x \qquad dv = \sin x \, dx$

June 11, 2015   13 / 42

$$I = e^x \sin x + e^x \cos x - I \qquad \text{add } I \text{ to both sides}$$

$$+I \qquad\qquad\qquad +I$$

$$2I = e^x \sin x + e^x \cos x \qquad \text{Divide by 2 and tack on } +C$$

$$I = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$$

$$\boxed{\int e^x \cos x \, dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C}$$

Bonus Result:

$$I = e^x \sin x - \int e^x \sin x \, dx \quad \Rightarrow$$

$$\int e^x \sin x \, dx = e^x \sin x - I$$

$$= e^x \sin x - \left(\frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C \right)$$

$$= \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x - C$$

$$= \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + k$$

Definite integrals

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\, dx$$

Evaluate $\int_1^e \ln x\, dx$

Let $\quad u = \ln x \qquad du = \frac{1}{x}\, dx$

$\qquad\qquad\qquad v = x \qquad\quad dv = dx$

$$\int_1^e \ln x\, dx = x \ln x \Big|_1^e - \int_1^e x \cdot \frac{1}{x}\, dx$$

$$= x \ln x \Big|_1^e - \int_1^e dx$$

$$= x \ln x \Big|_1^e - x \Big|_1^e$$

$$= e \ln e - 1 \ln 1 - \left(e - 1 \right)$$

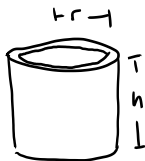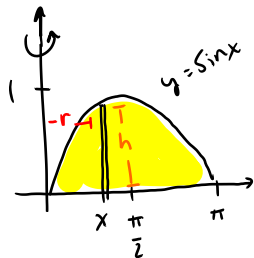$$= e \cdot 1 - 1 \cdot 0 - e + 1$$

$$= 1$$

* Not

$$\int \ln x \, dx = x \ln x - x + C$$

Example

The region between the curve $y = \sin x$ and the x-axis on the interval $[0, \pi]$ is rotated about the y-axis to form a solid. Find the volume of that solid.

Volume of one Shell is

$$2\pi r h \, \Delta x$$

$r = x$

$h = \sin x$

$$= 2\pi x \sin x \, \Delta x$$

To integrate in x we need to use Shells.

Total Volume

$$V = \int_0^\pi 2\pi x \sin x \, dx$$

Int. by parts

$$V = 2\pi \int_0^\pi x \sin x \, dx$$

$u = x \qquad du = dx$

$v = -\cos x \qquad dv = \sin x \, dx$

$$= 2\pi \left[-x \cos x \Big|_0^\pi - \int_0^\pi (-\cos x) \, dx \right]$$

$$= 2\pi \left[-x \cos x \Big|_0^\pi + \int_0^\pi \cos x \, dx \right]$$

$$= 2\pi \left[-x \cos x \Big|_0^\pi + \sin x \Big|_0^\pi \right]$$

$$= 2\pi \left[-\pi \cos\pi - (-0\cos 0) + \sin\pi - \sin 0 \right]$$

$$= 2\pi \left(-\pi(-1) + 0 \right)$$

$$= 2\pi^2$$

Section 7.2: Trigonometric Integrals

Compare the two integrals

$$\int \cos^3 x \, dx \qquad \text{and} \qquad \int (1 - \sin^2 x) \cos x \, dx$$

these
are
same
$\cos^2 x = 1 - \sin^2 x$
since

For the second one

set $u = \sin x \qquad du = \cos x \, dx$

$$\int (1 - \sin^2 x) \cos x \, dx = \int (1 - u^2) \, du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C$$

Evaluate $\int \sin^2 x \cos^3 x \, dx$

Use the same approach

$$= \int \sin^2 x \, \cos^2 x \, \cos x \, dx$$

$$= \int \sin^2 x \, (1 - \sin^2 x) \, \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$= \int u^2 (1 - u^2) \, du$$

$$= \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

$\int \sin^m x \cos^n x \, dx$

(a) If n is odd ($n = 2k + 1$), then save one cosine for du, write the remaining cosines as

$$\cos^{2k} x = (\cos^2 x)^k = (1 - \sin^2 x)^k,$$

and choose the substitution $u = \sin x$.

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$$

$$= \int u^m (1 - u^2)^k \, du$$

$\int \sin^m x \cos^n x \, dx$

(b) If m is odd ($m = 2p + 1$), then save one sine for du, write the remaining sines as

$$\sin^{2p} x = (\sin^2 x)^p = (1 - \cos^2 x)^p,$$

and choose the substitution $u = \cos x$.

$$\int \sin^m x \cos^n x \, dx = \int \sin x (1 - \cos^2 x)^p \cos^n x \, dx$$

$$= -\int u^n (1 - u^2)^p \, du$$

Evaluate

$$\int \sin^5 x \, dx$$

$$\sin^5 x = \sin^4 x \, \sin x$$

$$= \left(\sin^2 x \right)^2 \sin x$$

$$= \left(1 - \cos^2 x \right)^2 \sin x$$

$$= \left(1 - 2\cos^2 x + \cos^4 x \right) \sin x$$

$$\int \sin^5 x \, dx = \int \left(1 - 2\cos^2 x + \cos^4 x \right) \sin x \, dx$$

Let $u = \cos x$ $du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= -\int (1 - 2u^2 + u^4) \, du$$

$$= -\left(u - 2\frac{u^3}{3} + \frac{u^5}{5}\right) + C$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$