# June 12 Math 1190 sec. 51 Summer 2017

## Section 2.2: The Derivative as a Function

Recall that we defined the derivative of a function $f$ at the number $c$ by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

which can also be written as

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

We can interpret this in many ways

- the rate of change of $f$ at $c$,
- the slope of the line tangent to the graph of $f$ at $(c, f(c))$,
- velocity if $f$ is the position of a moving object.

# The Derivative Function

Let $f$ be a function. Define the new function $f'$ by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

called the **derivative** of $f$. The domain of this new function is the set

$$\{x | x \text{ is in the domain of } f, \text{ and } f'(x) \text{ exists}\}.$$

$f'$ is read as "$f$ prime."

## Example

Let $f(x) = \sqrt{x-1}$. Identify the domain of $f$. Find $f'$ and identify its domain.

For $x$ in the domain of $f$, we require $x-1 \geq 0$.

i.e. $x \geq 1$. In interval notation, the domain of $f$ is $[1, \infty)$.

By definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}$$

$$= \lim_{h \to 0} \left( \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \right) \left( \frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}} \right)$$

$$= \lim_{h \to 0} \frac{x+h-1 - (x-1)}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{x}+h-\cancel{1} - \cancel{x} + \cancel{1}}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-1} + \sqrt{x-1}}$$

$$= \frac{1}{\sqrt{x+0-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

So $\quad f'(x) = \dfrac{1}{2\sqrt{x-1}}.$

For $x$ in the domain of $f'$, we require $x-1>0$.

In interval notation, the domain of $f'$ is

$$(1, \infty).$$

Use the result to find $f'(5)$ where $f(x) = \sqrt{x-1}$.

Because we know that $f'(x) = \dfrac{1}{2\sqrt{x-1}}$,

we can find $f'(5)$ by evaluating $f'(x)$

@ $x = 5$.

$$f'(5) = \frac{1}{2\sqrt{5-1}} = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

Find the equation of the line tangent to $f$ at $(5, f(5))$.

$f(x) = \sqrt{x-1}$ , so $f(5) = \sqrt{5-1} = \sqrt{4} = 2$.

So our point $(5, f(5)) = (5, 2)$.

Also, the slope $m_{tan} = f'(5)$. So $m_{tan} = f'(5) = \frac{1}{4}$.

pt. slope

$y - y_0 = m(x - x_0)$

$y - 2 = \frac{1}{4}(x-5)$

$y - 2 = \frac{1}{4}x - \frac{5}{4}$

$y = \frac{1}{4}x - \frac{5}{4} + 2 \Rightarrow \boxed{y = \frac{1}{4}x + \frac{3}{4}}$

# Question

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let $f(x) = 2x^2 + x$; determine $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 + (x+h) - (2x^2+x)}{h}$$

(a) $f'(x) = 4$

(b) $f'(x) = 2x + 1$

(c) $f'(x) = 4x + x$

(d) $f'(x) = 4x + 1$

# How are the functions $f(x)$ and $f'(x)$ related?

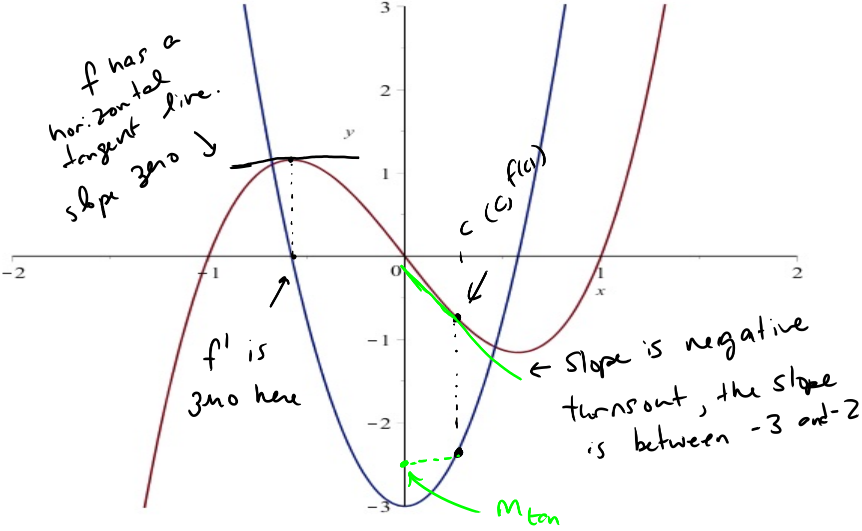



Figure: Red $f(x)$, Blue $f'(x)$

# Remarks:

- if $f(x)$ is a function of $x$, then $f'(x)$ is a new function of $x$ (called the derivative of $f$)
- The number $f'(c)$ (if it exists) is the slope of the curve of $y = f(x)$ at the point $(c, f(c))$
- this is also the slope of the tangent line to the curve of $y$ at $(c, f(c))$
- "slope of the curve", "slope of the tangent line", and "rate of change" are the same concept

**Definition:** A function $f$ is said to be *differentiable* at $c$ if $f'(c)$ exists. It is called *differentiable* on an open interval $I$ if it is differentiable at each point in $I$.

# Failure to be Differentiable

We saw that the domain of $f(x) = \sqrt{x-1}$ is $[1, \infty)$ whereas the domain of its derivative $f'(x) = \frac{1}{2\sqrt{x-1}}$ was $(1, \infty)$. Hence $f$ **is not differentiable at** 1.

**Another Example:** Show that $y = |x|$ is not differentiable at zero.

For $f(x) = |x|$, $f'(0)$ would be

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h\to 0} \frac{|h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

Recall

$$|h| = \begin{cases} h, & h \geq 0 \\ -h, & h < 0 \end{cases}$$

Using one sided limits

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

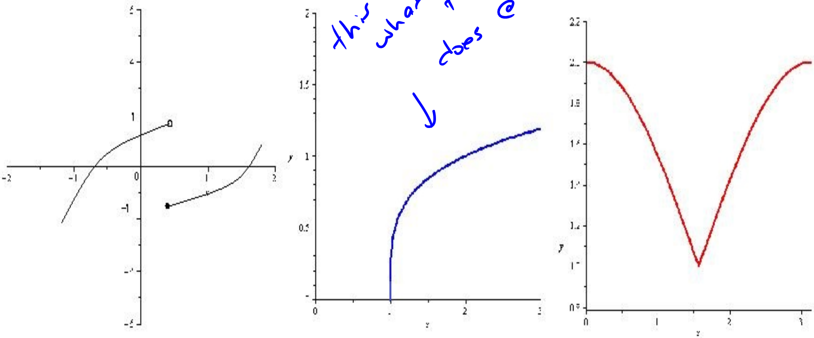
$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

so $\displaystyle\lim_{h\to 0} \frac{|h|}{h}$ DNE

That is, $f'(0)$ DNE. $y = |x|$ is not differentiable @ zero.

# Failure to be differentiable: Discontinuity, Vertical tangent, or Corner/Cusp

this is what $f(x) = \sqrt{x-1}$ does @ 1



## Theorem

Differentiability implies continuity.

That is, if $f$ is differentiable at $c$, then $f$ is continuous at $c$. Note that the corner example shows that the converse of this is not true!

for example $f(x) = |x|$ is continuous @ 0, but it's not differentiable there.

# Questions

(1) **True or False:** Suppose that we know that $f'(3) = 2$. We can conclude that $f$ is continuous at 3.

If it's differentiable @ 3, it must be continuous @ 3.

(2) **True or False:** Suppose that we know that $f'(1)$ does not exist. We can conclude that $f$ is discontinuous at 1.

It may or may not be continuous, we'd need more information to reach a conclusion.

# Section 2.3: The Derivative of a Polynomial; The Derivative of $e^x$

First some notation:

If $y = f(x)$, the following notation are interchangeable:

$$f'(x) = y'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

$$\text{Leibniz Notation:} \qquad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

You can think of $\quad D, \text{ or } \dfrac{d}{dx} \quad$ as an "operator."

It acts on a function to produce a new function—its derivative.
Taking a derivative is referred to as *differentiation*.

# Some Derivative Rules

The derivative of a constant function is zero.

$$\frac{d}{dx}c = 0$$

The derivative of the identity function is one.

this reads as
" the derivative of x with respect to x is one "

$$\frac{d}{dx}x = 1$$

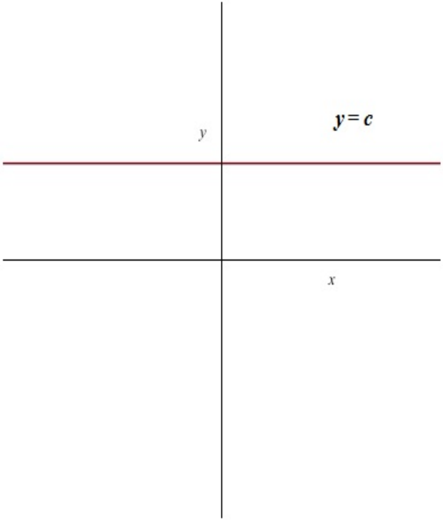
For positive integer $n$[1],

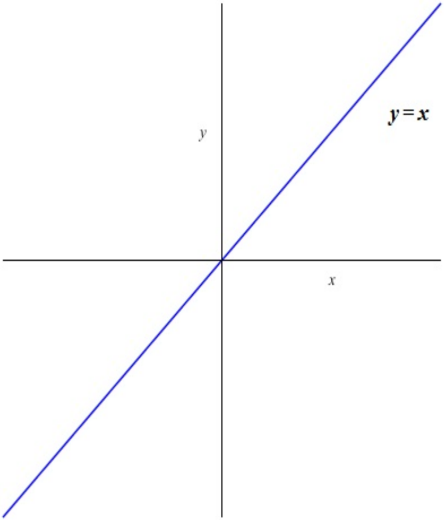
$$\frac{d}{dx}x^n = nx^{n-1}$$

This last one is called the **power rule.**

[1] This rule turns out to hold for any real number $n$, though the proofs for more general cases require results yet to come.

$\frac{d}{dx} c = 0, \qquad \frac{d}{dx} x = 1$



## Evaluate Each Derivative

(a) $\frac{d}{dx}(-7) = 0$ Der. of a constant

(b) $\frac{d}{dx} 3\pi = 0$ D.H.0

(c) $\frac{d}{dx} x^9 = 9x^8$ power rule $\frac{d}{dx} x^n = n x^{n-1}$

# More Derivative Rules

Assume $f$ and $g$ are differentiable functions and $k$ is a constant.

$$\text{Constant multiple rule:} \quad \frac{d}{dx}kf(x) = kf'(x)$$

$$\text{Sum rule:} \quad \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

$$\text{Difference rule:} \quad \frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

The rules we have thus far allow us to find the derivative of any polynomial function.

# Example: Evaluate Each Derivative

(a) $\frac{d}{dx}(x^4 - 3x^2) = \frac{d}{dx}x^4 - \frac{d}{dx}3x^2$ (Difference)

$= \frac{d}{dx}x^4 - 3\frac{d}{dx}x^2$ (constant factor)

$= 4x^{4-1} - 3(2x^{2-1})$ (power rule)

$\frac{d}{dx}x^n = nx^{n-1}$

$= 4x^3 - 3(2x)$

$= 4x^3 - 6x$

(b) $\frac{d}{dx}(2x^3+3x^2-12x+1) = \frac{d}{dx}2x^3 + \frac{d}{dx}3x^2 - \frac{d}{dx}12x + \frac{d}{dx}1$

$= 2\frac{d}{dx}x^3 + 3\frac{d}{dx}x^2 - 12\frac{d}{dx}x + \frac{d}{dx}1$

$= 2\left(3x^{3-1}\right) + 3\left(2x^{2-1}\right) - 12\cdot 1 + 0$

$= 2(3x^2) + 3(2x) - 12$

$= 6x^2 + 6x - 12$

# Example

If $f(x) = 2x^3 + 3x^2 - 12x + 1$, find all points on the graph of $f$ at which the slope of the graph is zero.

If the slope of the graph @ $(c, f(c))$ is zero, then $f'(c) = 0$. So this can be restated as saying "find all values of $c$ at which $f'(c) = 0$.".

We know $f'(x) = 6x^2 + 6x - 12$. So

$$f'(c) = 0 \Rightarrow 6c^2 + 6c - 12 = 0$$

Solve this:

$$6(c^2 + c - 2) = 0$$

$$6(c+2)(c-1) = 0$$

This holds if $\quad c+2=0 \quad$ or $\quad c-1=0$

$c=-2 \quad$ or $\quad c=1$

The points have x-values $-2$ and $1$. We get the y's from $f(x)=2x^3+3x^2-12x+1$

$$f(-2)=2(-2)^3+3(-2)^2-12(-2)+1=2(-8)+3(4)+24+1=21$$

$$f(1)=2(1)^3+3(1)^2-12(1)+1=2+3-12+1=-6$$

There are two points where the slope of $f$ is zero, $(1,-6)$ and $(-2,21)$.

# The Derivative of $e^x$

Consider $a > 0$ and $a \neq 1$. Let $f(x) = a^x$. Analyze the limit $f'(0)$ and $f'(x)$

By definition

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{a^{0+h} - a^0}{h}$$

$$= \lim_{h \to 0} \frac{a^h - 1}{h}$$

$$\Rightarrow \boxed{f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h} \text{ provided this limit exists.}}$$

This limit does exist; it depends on $a$ and is some number.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \qquad a^{x+h} = a^x \cdot a^h$$

$$= \lim_{h\to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h\to 0} \frac{(a^h - 1)\, a^x}{h} = \left(\lim_{h\to 0} \frac{a^h - 1}{h}\right) a^x$$

The red expression is $f'(0)$ - some number.

So

$$f'(x) = f'(0)\, a^x$$

a constant times $a^x$

# The Derivative of $e^x$

**Definition:** The number $e$ is defined[2] by the property

$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$

It follows that

**Theorem:** $y=e^x$ is differentiable (at all real numbers) and

$$\frac{d}{dx}e^x=e^x.$$

[2]This is one of several mutually consistent ways to defined this number. Numerically, $e\approx 2.718282$.

# Question

Evaluate the derivative of $f(x) = 4x^6 - 2e^x$

(a) $f'(x) = 24x^5 - 2xe^{x-1}$

(b) $f'(x) = 6x^5 - e^x$

(c) $f'(x) = 24x^5 - 2e^{x-1}$

(d) $f'(x) = 24x^5 - 2e^x$

# Section 2.4: Differentiating a Product or Quotient; Higher Order Derivatives

**Motivating Example:** Evaluate the derivative

$$\frac{d}{dx}[x^3(2x^2-6x+17)]$$

we don't need new rules, we can distribute first.

$$\frac{d}{dx}\left[2x^5 - 6x^4 + 17x^3\right]$$

$$= 2(5x^4) - 6(4x^3) + 17(3x^2)$$

$$= 10x^4 - 24x^3 + 51x^2$$

# Derivative of A Product

Now consider evaluating the derivative

$$\frac{d}{dx}[(3x^5-2x^2+x)(x^3-2x^2+x-1)]$$

As before, we can expand this all out to get an 8th degree polynomial.

This requires a lot of algebra before even starting to take the derivative.

We'd like a way of taking the derivative of a product.

# Derivative of A Product

**Theorem: (Product Rule)** Let $f$ and $g$ be differentiable functions of $x$. Then the product $f(x)g(x)$ is differentiable. Moreover

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

This can be stated using Leibniz notation as

$$\frac{d}{dx}[f(x)g(x)] = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}.$$

# Example

Compute $\frac{d}{dx}x^5$ using the product rule with $f(x) = x^2$ and $g(x) = x^3$. Compare this with the result from the power rule on $x^5$.

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + f(x)\,g'(x)$$

If $f(x) = x^2$, then $f'(x) = 2x$

and $g(x) = x^3$, then $g'(x) = 3x^2$

So $$\frac{d}{dx}\left[x^5\right] = \frac{d}{dx}\left[x^2 \cdot x^3\right] = \overset{f'}{2x}\,(\overset{g}{x^3}) + \overset{f}{x^2}\,(\overset{g'}{3x^2})$$

$$= 2x^4 + 3x^4 = 5x^4$$

Note this matches the power rule

$$\frac{d}{dx}x^5 = 5x^4$$

# Example

Evaluate $\dfrac{d}{dx}[(3x^5-2x^2+x)(x^3-2x^2+x-1)]$

If $f(x) = 3x^5 - 2x^2 + x$, $f'(x) = 15x^4 - 4x + 1$

If $g(x) = x^3 - 2x^2 + x - 1$, $g'(x) = 3x^2 - 4x + 1$

$$\frac{d}{dx}\left[(3x^5-2x^2+x)\left(x^3-2x^2+x-1\right)\right] =$$

$$\underbrace{(15x^4-4x+1)}_{f'}\underbrace{(x^3-2x^2+x-1)}_{g} + \underbrace{(3x^5-2x^2+x)}_{f}\underbrace{(3x^2-4x+1)}_{g'}$$

# Example

Evaluate $\frac{d}{dx}e^{2x}$ using the product rule.

By properties of exponentials $e^{2x} = e^x \cdot e^x$

$$\frac{d}{dx}\left[e^{2x}\right] = \frac{d}{dx}\left[e^x \cdot e^x\right] = \left(\frac{d}{dx}e^x\right)\cdot e^x + e^x\left(\frac{d}{dx}e^x\right)$$

$$= e^x \cdot e^x + e^x \cdot e^x$$

$$= e^{2x} + e^{2x} \qquad = 2e^{2x}$$

# Question

$$\frac{d}{dx}[f(x)g(x)] = f'(x)\,g(x) + f(x)g'(x)$$

Evaluate $f'(x)$ where $f(x) = 3x^4e^{2x}$.

$$f'(x) = \left(\frac{d}{dx}3x^4\right)e^{2x} + 3x^4\left(\frac{d}{dx}e^{2x}\right)$$

(a) $f'(x) = 6x^4e^{2x}$

(b) $f'(x) = 12x^3e^{2x} + 6x^4e^{2x}$

(c) $f'(x) = 24x^3e^{2x}$

(d) $f'(x) = 3x^4e^{2x} + 12x^3e^{2x}$

## The Derivative of a Quotient

**Theorem (Quotient Rule)** Let $f$ and $g$ be differentiable functions of $x$. Then on any interval for which $g(x) \neq 0$, the ratio $\frac{f(x)}{g(x)}$ is differentiable. Moreover

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

This can be stated using Leibniz notation as

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{[g(x)]^2}.$$

An immediate consequence of this is that

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}.$$

# Example

Use the quotient rule to show that for positive integer $n$[3]

$$\frac{d}{dx}x^{-n} = -nx^{-n-1}$$

For $n$ a positive integer, $x^{-n} = \frac{1}{x^n}$

Use $\frac{d}{dx}\left[\frac{1}{g(x)}\right] = \frac{-g'(x)}{(g(x))^2}$ where $g(x) = x^n$

$$\frac{d}{dx}\left[x^{-n}\right] = \frac{d}{dx}\left[\frac{1}{x^n}\right] = \frac{-n\,x^{n-1}}{(x^n)^2} = \frac{-n\,x^{n-1}}{x^{2n}}$$

[3] Note that this shows that the power rule works for both positive and negative integers.

$$= -n\,x^{n-1-2n}$$

$$= -n\,x^{-n-1}$$

That is, $\frac{d}{dx}\left[x^{-n}\right] = -n\,x^{-n-1}$

For example, $\frac{d}{dx}x^{-5} = -5\,x^{-6}$

# Example

Evaluate $\dfrac{d}{dx}e^{-x}$

We'll use $e^{-x} = \dfrac{1}{e^x}$

use $\dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = \dfrac{-g'(x)}{(g(x))^2}$ with $g(x) = e^x$

$$\frac{d}{dx}e^{-x} = \frac{d}{dx}\left[\frac{1}{e^x}\right] = \frac{-e^x}{(e^x)^2} = \frac{-e^x}{e^{2x}}$$

$$= -e^{x-2x} = -e^{-x}$$

# Example

Evaluate $\dfrac{d}{dx}\left(\dfrac{e^x}{x^2+2x}\right)$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$f(x) = e^x$, $f'(x) = e^x$

$g(x) = x^2 + 2x$

$g'(x) = 2x + 2$

$$= \frac{e^x(x^2+2x) - e^x(2x+2)}{(x^2+2x)^2}$$

$$= \frac{x^2e^x + 2xe^x - 2xe^x - 2e^x}{(x^2+2x)^2} = \frac{x^2e^x - 2e^x}{(x^2+2x)^2} = \frac{(x^2-2)e^x}{(x^2+2x)^2}$$

# Question

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Evaluate $f'(x)$ where $\quad f(x) = \dfrac{3x+4}{x^2+1}$

(a) $\quad f'(x) = \dfrac{3x^2+8x-3}{(x^2+1)^2}$

(b) $\quad f'(x) = \dfrac{3-2x(3x+4)}{(x^2+1)}$

(c) $\quad f'(x) = \dfrac{-3x^2-8x+3}{(x^2+1)^2}$

(d) $\quad f'(x) = \dfrac{-3x^2-8x+3}{x^4+1}$

# Higher Order Derivatives:

Given $y = f(x)$, the function $f'$ may be differentiable as well. We may take its derivative which is called the **second derivative** of $f$. We use the following notation and language:

First derivative: $\dfrac{dy}{dx} = y' = f'(x)$

Second derivative: $\dfrac{d}{dx}\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2} = y'' = f''(x)$

Third derivative: $\dfrac{d}{dx}\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = y''' = f'''(x)$

Fourth derivative: $\dfrac{d}{dx}\dfrac{d^3y}{dx^3} = \dfrac{d^4y}{dx^4} = y^{(4)} = f^{(4)}(x)$

$n^{th}$ derivative: $\dfrac{d}{dx}\dfrac{d^{n-1}y}{dx^{n-1}} = \dfrac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x)$

# Remarks on Notation

- $\frac{d}{dx}$ can *operate* on a function to produce a new function; e.g.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

- It's too hard to read multiple primes (say beyond 3). Parentheses **must** be used to distinguish powers from derivatives.

$$y^5 \text{ is the fifth power of } y;$$

$$y^{(5)} \text{ is the fifth derivative of } y$$