

Section 7.2: Trigonometric Integrals

$$\int \sin^m x \cos^n x \, dx$$

- (a) If n is odd ($n = 2k + 1$), then save one cosine for du , write the remaining cosines as

$$\cos^{2k} x = (\cos^2 x)^k = (1 - \sin^2 x)^k,$$

and choose the substitution $u = \sin x$.

$$\begin{aligned}\int \sin^m x \cos^n x \, dx &= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx \\ &= \int u^m (1 - u^2)^k \, du\end{aligned}$$

$$\int \sin^m x \cos^n x \, dx$$

(b) If m is odd ($m = 2p + 1$), then save one sine for du , write the remaining sines as

$$\sin^{2p} x = (\sin^2 x)^p = (1 - \cos^2 x)^p,$$

and choose the substitution $u = \cos x$.

$$\begin{aligned}\int \sin^m x \cos^n x \, dx &= \int \sin x (1 - \cos^2 x)^p \cos^n x \, dx \\ &= - \int u^n (1 - u^2)^p \, du\end{aligned}$$

Evaluate

$$\int \sin^3 x \cos^3 x \, dx$$

rewrite the integrand:

$$\begin{aligned}\sin^3 x \cos^3 x &= \sin^3 x \cos^2 x \cos x \\&= \sin^3 x (1 - \sin^2 x) \cos x \\&= (\sin^3 x - \sin^5 x) \cos x\end{aligned}$$

$$\int \sin^3 x \cos^3 x \, dx = \int (\sin^3 x - \sin^5 x) \cos x \, dx$$

$$\text{Let } u = \sin x \quad du = \cos x \, dx$$

$$= \int (u^3 - u^5) \, du$$

$$= \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$$

Alternative approach:

$$\cos^3 x \sin^3 x = \cos^3 x \sin^2 x \sin x$$

$$= \cos^3 x (1 - \cos^2 x) \sin x$$

$$= (\cos^3 x - \cos^5 x) \sin x$$

$$\int \cos^3 x \sin^3 x dx = \int (\cos^3 x - \cos^5 x) \sin x dx$$

Let $u = \cos x \quad du = -\sin x dx$

$$-du = \sin x dx$$

$$= - \int (u^3 - u^5) du$$

$$= -\left(\frac{u^4}{4} - \frac{u^6}{6}\right) + K = -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + K$$

Evaluate

$$\int \sqrt{\sin x} \cos^5 x \, dx$$

$$\begin{aligned} (\sin x)^{\frac{1}{2}} \cos^5 x &= (\sin x) \cos^4 x \cos x \\ &= (\sin x) (\cos^2 x)^2 \cos x \\ &= (\sin x)^{\frac{1}{2}} (1 - \sin^2 x)^2 \cos x \\ &= (\sin x)^{\frac{1}{2}} (1 - 2\sin^2 x + \sin^4 x) \cos x \end{aligned}$$

$$\int \sqrt{\sin x} \cos^5 x \, dx = \int (\sin x)^{\frac{1}{2}} (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$$

$$u = \sin x \quad du = \cos x \, dx$$

$$= \int u^{\frac{1}{2}} (1 - 2u^2 + u^4) \, du$$

$$= \int (u^{\frac{1}{2}} - 2u^{\frac{5}{2}} + u^{\frac{9}{2}}) \, du$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - 2 \frac{u^{\frac{7}{2}}}{\frac{7}{2}} + \frac{u^{\frac{11}{2}}}{\frac{11}{2}} + C$$

$$= \frac{2}{3} (\sin x)^{\frac{3}{2}} - \frac{4}{7} (\sin x)^{\frac{7}{2}} + \frac{2}{11} (\sin x)^{\frac{11}{2}} + C$$

What if both m and n are even?

Recall the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

$$\int \sin^m x \cos^n x \, dx$$

(c) If both m and n are even, use the half-angle identities above. The identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

may also be useful.

Evaluate

$$\int \cos^4 \theta d\theta$$

rewrite the integrand using the
half angle ID

$$\begin{aligned}\cos^4 \theta &= \cos^2 \theta \cos^2 \theta \\&= \frac{1}{2}(1 + \cos 2\theta) \frac{1}{2}(1 + \cos 2\theta) \\&= \frac{1}{4}(1 + \cos 2\theta)(1 + \cos 2\theta) \\&= \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta) \\&= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\right)\end{aligned}$$

$$= \frac{1}{4} (1 + 2 \cos 2\theta + \frac{1}{2} + \frac{1}{2} \cos 4\theta)$$

$$= \frac{1}{4} \left(\frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right)$$

$$\int \cos^4 \theta d\theta = \int \left(\frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta \right) d\theta$$

$$= \frac{3}{8} \theta + \frac{1}{2} \cdot \frac{1}{2} \sin 2\theta + \frac{1}{8} \cdot \frac{1}{4} \sin 4\theta + C$$

$$= \frac{3}{8} \theta + \frac{1}{4} \sin 2\theta + \frac{1}{32} \sin 4\theta + C$$

Other Uses of Trig Identities

Recall:

$$\frac{d}{dx} \tan x = \sec^2 x \quad \text{and} \quad \frac{d}{dx} \sec x = \sec x \tan x$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\int \tan x \, dx = \ln |\sec x| + C$$

and

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

If $u = \tan x$, we need

$$du = \sec^2 x \, dx$$

+ an even number of lefts
over seconds.

If $u = \sec x$, we need

$$du = \sec x \tan x \, dx + \text{an}$$

even number of tangents.

Evaluate

$$\text{For } u = \tan \theta \quad du = \sec^2 \theta d\theta$$

$$\text{For } u = \sec \theta \quad du = \sec \theta \tan \theta d\theta$$

$$\int \sec^4 \theta \tan \theta d\theta$$

Either will work:
will do both

Solution 1:

$$\sec^4 \theta \tan \theta = \sec^3 \theta \sec \theta \tan \theta$$

$$\int \sec^4 \theta d\theta = \int \sec^3 \theta \sec \theta \tan \theta d\theta$$

$$\text{Let } u = \sec \theta$$

$$du = \sec \theta \tan \theta d\theta$$

$$= \int u^3 du$$

$$= \frac{u^4}{4} + C = \frac{\sec^4 \theta}{4} + C$$

Solution 2:

$$\begin{aligned}\sec^4 \theta \tan \theta &= \sec^2 \theta \tan \theta \sec^2 \theta \\ &= (\tan^2 \theta + 1) \tan \theta \sec^2 \theta\end{aligned}$$

$$= (\tan^3 \theta + \tan \theta) \sec^2 \theta d\theta$$

$$\int \sec^4 \theta \tan \theta d\theta = \int (\tan^3 \theta + \tan \theta) \sec^2 \theta d\theta$$

$$\text{Let } u = \tan \theta$$

$$du = \sec^2 \theta d\theta$$

$$= \int (u^3 + u) du$$

$$= \frac{u^4}{4} + \frac{u^2}{2} + k = \frac{\tan^4 \theta}{4} + \frac{\tan^2 \theta}{2} + k$$

Evaluate

$$\int \tan^3 t dt$$

If $u = \tan t$ $du = \sec^2 t dt$

$u = \sec t$ $du = \sec t \tan t dt$

$$\tan^3 t = \tan t \tan^2 t$$

$$= \tan t (\sec^2 t - 1)$$

$$= \tan t \sec^2 t - \tan t$$

$$\int \tan^3 t dt = \int (\tan t \sec^2 t - \tan t) dt$$

$$= \int \tan t \sec^2 t dt - \int \tan t dt$$

Let $u = \tan t$

$$du = \sec^2 t dt \quad = \int u du - \int \tan t dt$$

$$= \frac{u^2}{2} - \ln |\sec t| + C$$

$$= \frac{\tan^2 t}{2} - \ln |\sec t| + C$$

Recall:

$$\frac{d}{dx} \cot x = -\csc^2 x \quad \frac{d}{dx} \csc x = -\csc x \cot x$$

$$\cot^2 x + 1 = \csc^2 x$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

$$\int \cot x \, dx = -\ln|\csc x| + C = \ln|\sin x| + C$$